Letters in Mathematical Physics (2023) 113:29
https://doi.org/10.1007/s11005-023-01654-2

Rigidity of compact static near-horizon geometries with negative cosmological constant

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Received: 19 January 2023 / Revised: 14 February 2023 / Accepted: 21 February 2023 /
Published online: 4 March 2023

Abstract
In this note, we show that compact static near-horizon geometries with negative cosmological constant are either Einstein or the product of a circle and an Einstein metric. Chruściel, Reall, and Todd proved rigidity when the cosmological constant vanishes, in which case one get the stronger result that the space is Ricci flat (Chruściel et al. in Class Quantum Gravity 23:549–554, 2006). It has been previously asserted that a stronger rigidity statement also holds for negative cosmological constant, but Bahuaud, Gunasekaran, Kunduri, and Woolgar recently pointed out that this was not the case (Bahuaud et al. in Lett Math Phys 112(6):116, 2022). They showed, moreover, that for a compact static near-horizon geometry with negative cosmological constant, the potential vector field X is constant length and divergence-free. We give an argument using the Bochner formula to improve their conclusion to X being a parallel field, which implies the optimal rigidity result. The result also holds more generally for m-Quasi Einstein metrics with $m > 0$.

Keywords Quasi-Einstein manifold · Near horizon geometry · Rigidity

MSC codes 53C24 · 53C25 · 83C05 · 83C57

1 Introduction

In this paper, an m-quasi Einstein metric is a Riemannian manifold (M, g) along with a vector field X such that

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m} X^* \otimes X^* = \lambda g$$

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where Ric is the Ricci tensor of g, $\mathcal{L}_X g$ is the Lie derivative of the metric, X^* is the dual one-form to X using the metric g, and m and λ are constants. In a slight abuse of notation, following [1], we will call X exact or closed depending on whether the corresponding one form X^* is exact or closed. In terms of the vector field X, this implies that X is exact (or gradient) if $X = \nabla f$ for some real valued function f defined on M and X is closed if around every point X is locally the gradient of a function defined on an open neighborhood around the point.

There are many variations of the definition of an m-quasi Einstein metric that may replace λ or m by functions or require X to be exact. We believe that in the first time the term quasi-Einstein was used for this equation was in [4] where it is assumed that λ and m were constants with $m > 0$ and X exact. One idea that has inspired investigation of the m-quasi Einstein equation is the similarity to the Ricci soliton equation, $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g$, which formally one can think of as the case where $m \to \infty$. Ricci solitons describe geometric fixed points of the Ricci flow that change only by diffeomorphism and scaling and they are an important aspect of the study of singularity formulation in the Ricci flow.

In fact, an early result on the study of Ricci solitons is directly relevant to our considerations here: Ivey showed in [9] that a compact Ricci soliton, $\mathrm{Ric}+\frac{1}{2}\mathcal{L}_X g = \lambda g$ with $\lambda \leq 0$ must be an Einstein metric with $X = 0$. On the one hand, Ivey's result is not true when the parameter m is introduced as, for example, Chen–Liang–Zhu constructed left-invariant metrics on simple groups, which are m-quasi Einstein for all possible values of $m > 0$ and λ [5]. Lim also later classified completely the locally homogeneous three-dimensional examples [12].

On the other hand, when X is exact, Kim-Kim showed that the Ivey result does hold when $\lambda \leq 0$ [10]. Moreover, when X is closed and $\lambda = 0$, Chruściel–Reall–Todd proved X must be zero [6]. They also stated that the result holds when $\lambda < 0$. However, Bahuaud–Gunasekaran–Kunduri–Woolgar observed that the product of a circle with an Einstein metric and X a parallel field on the S^1 factor of length $\sqrt{-m\lambda}$ gives a counter-example [1]. Note that clearly this example is closed but not exact. They also show in this case that X must be a divergence-free vector of length $\sqrt{-m\lambda}$. The purpose of this note is to complete the investigation of the case where $m > 0$, $\lambda < 0$, and X is closed by bridging the gap between the example and rigidity statement in [1].

Theorem 1.1 *Suppose (M^n, g_M, X) is a compact m-quasi Einstein metric with closed X and $\lambda < 0$. Then, either*

(1) $X = 0$ *and* $\mathrm{Ric}_{g_M} = \lambda g_M$, *or*
(2) (M, g) *is isometric to a product metric* $(S^1 \times N, d\theta^2 + g_N)$ *where* (N, g_N) *is an* $(n-1)$*-dimensional Einstein metric* $\mathrm{Ric}_{g_N} = \lambda g_N$, *and* $X = \pm\sqrt{-m\lambda}\frac{\partial}{\partial\theta}$.

When $m = 2$, the m-quasi Einstein metric is also called a vacuum near-horizon geometry (with λ related to the cosmological constant by a positive factor). These metrics arise from a limiting procedure from extreme black hole spacetime metrics, which have dimension $n + 2$. If X is closed, it is called a static vacuum near-horizon geometry. This was the motivation in [1, 6] for studying this equation in the closed case. See [1] for more information and references in this direction. As is pointed out in

[1], when $m = 2$ the examples in part (2) of Theorem 1.1 do arise as the near-horizon geometry of the $(n+2)$-dimensional space-time, $g_{XBTZ} + g_N$, where g_{XBTZ} is an extreme BTZ black hole metric, see [1] for further details.

The limiting procedure of taking extreme black hole spacetimes to near-horizon geometries gives a number of examples of m-quasi Einstein metrics that are also interesting from the perspective of Riemannian geometry. For example, the near-horizon geometry of the extremal Kerr space time gives a (non-closed) 2-quasi Einstein metric on S^2 with $\lambda = 0$. This metric is, in a sense, a Bakry–Emery Ricci flat metric on the 2-sphere! See [11, Theorem 4.3] for a discussion of this metric, where it is shown that it is the unique such rotationally symmetric metric. It is an open question whether this is the only such metric on the 2-sphere. The corresponding uniqueness question when $\lambda < 0$ for the near-horizon geometry of extreme AdS–Kerr blackholes is also an interesting open question.

2 Proof of the Theorem

The idea of the proof is to show that X is a parallel field. One motivation for this comes from considering the locally homogeneous case where Chen–Liang–Zhu [5] and Lim [12] show that for any compact m-quasi Einstein metric, X must be a Killing field. If X is Killing, then ∇X is anti-symmetric. If X is closed, then ∇X is also symmetric, so this shows that a closed m-quasi Einstein metric on a compact locally homogeneous manifold has X parallel.

In the general case, we continue the argument of [1]. In their Theorem 1.2 (iii), they show under our assumptions that

$$\mathrm{div} X = 0 \quad \text{and} \quad |X|^2 = -m\lambda. \tag{2.1}$$

Let $p \in M$. Then, around p we have $X = \nabla f$ for some function f. Consider the Bochner formula applied to f:

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess} f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla f, \nabla \Delta f).$$

Since $\Delta f = \mathrm{div}(\nabla f)$, from (2.1), we obtain that

$$|\nabla X|^2 = -\mathrm{Ric}(X, X). \tag{2.2}$$

On the other hand, (2.1) also implies that

$$\mathcal{L}_X g(X, X) = 2\, g(\nabla_X X, X) = D_X|X|^2 = 0.$$

So the m-quasi Einstein equation gives that

$$\begin{aligned}\mathrm{Ric}(X, X) &= -\frac{1}{2}\mathcal{L}_X g(X, X) + \frac{|X|^4}{m} + \lambda|X|^2 \\ &= 0.\end{aligned}$$

Combined with (2.2), this shows that X is parallel, which gives a local isometric splitting of (M, g).

To see that the splitting must be global, let $(\widetilde{M}, \widetilde{g})$ be the universal covering of M with $\widetilde{g}$ the pullback of the metric g under the covering map and $\widetilde{X}$ the pullback of X. Then, we have the global splitting $(\widetilde{M}, \widetilde{g}) = (\mathbb{R} \times \widetilde{g}_N)$, $\widetilde{X} = \pm{-m\lambda}\frac{\partial}{\partial r}$. (M, g) is then a quotient of $(\widetilde{M}, \widetilde{g})$ by a group of isometries Γ. Since $(\widetilde{M}, \widetilde{g})$ is locally isometric to (M, g), it is also a m-quasi Einstein manifold, with vector field $\widetilde{X}$. Moreover, as $\widetilde{X}$ is a parallel field, $L_{\widetilde{X}}\widetilde{g} = 0$. So if $\widetilde{Y} \perp \widetilde{X}$, then the m-quasi Einstein equation gives

$$\mathrm{Ric}_{\widetilde{g}}(\widetilde{Y}, \widetilde{Y}) = \lambda|\widetilde{Y}|^2.$$

In particular, since $\lambda \neq 0$, $\mathrm{Ric}_{\widetilde{g}}(\widetilde{X}, \widetilde{X}) = 0$, and isometries must preserve the eigenspaces of the Ricci tensor, this shows that the isometry group of $(\widetilde{M}, \widetilde{g})$ splits as a product of the isometry group of $\mathbb{R}$ and $g_{\widetilde{N}}$. This then implies that the quotient (M, g) splits as a product $S^1 \times N$.

3 Some comments on the positive cosmological constant case

Perelman [13] also showed that for any compact Ricci soliton with $\lambda > 0$, one can add a Killing field to X so that X is a gradient field. In [1], an analogous result for the $m, \lambda > 0$ case is proven: a compact m-quasi Einstein metric with X closed is exact. The proofs, however, are quite different and, in the $m > 0$ case, the assumption that X be closed is necessary as the Kerr horizon metrics show. Note that on a complete manifold, if a vector field X satisfies $\frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^* \otimes X^* = 0$ then $X = 0$ by [12, Proposition 6.2]. Therefore, for m-quasi Einstein metrics with $m \neq \infty$, there is no way to add to the vector field X and preserve the equation, as one can do with Killing fields in the Ricci soliton case.

Since 2-dimensional compact m-quasi Einstein metrics with X exact have been classified [1, 2, 7], the most natural open case to consider is 3-dimensional compact m-quasi Einstein metrics with X exact. Exact m-quasi Einstein metrics also correspond with $(n + m)$-dimensional warped product Einstein metrics. See, for example, [2, 4, 7, 10] for references in this direction. In particular, warped product Einstein metrics produced by Böhm on $S^3 \times S^2$ give 2-quasi Einstein metrics on S^3, which are not constant curvature [3]. In this case, we also see that the space of quasi-Einstein metrics is more rich than that of Ricci solitons, since Ivey proved any compact 3-dimensional Ricci soliton is a space of constant curvature [9]. On the other hand, since m-quasi Einstein metrics with $\lambda > 0$ must have finite fundamental group, in dimension 3 their universal cover is diffeomorphic to S^3.

To our knowledge, the Böhm metrics are the only known examples of compact 3-dimensional 2-quasi Einstein metrics with X exact. These metrics are rotationally symmetric. In [7], it is shown that locally conformally flat compact m-quasi Einstein metrics with exact X are rotationally symmetric. A natural candidate for more examples that are not locally conformally flat could be doubly warped product metrics on S^3, see [7, Example 3.5]. We note that the $\lambda = 0$ case (and X not assumed closed) of doubly warped products is addressed in [8]. It also appears to be an open question whether the Böhm metrics can arise as the near-horizon limit of a "reasonable" 5-dimensional static spacetime such as one that is asymptotically flat or AdS.

Acknowledgements The author is supported by NSF grant #1654034 and would like to thank Eric Woolgar for making him aware of this problem and for helpful feedback in preparing this note. We also thank the referee for very helpful feedback.

Data availability Data sharing was not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The author has no conflicts of interest.

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