Calc. Var. (2018) 57:141
https://doi.org/10.1007/s00526-018-1415-x

Calculus of Variations

Gradient shrinking Ricci solitons of half harmonic Weyl curvature

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Received: 11 October 2017 / Accepted: 28 July 2018 / Published online: 29 August 2018

Abstract
Gradient Ricci solitons and metrics with half harmonic Weyl curvature are two natural generalizations of Einstein metrics on four-manifolds. In this paper we prove that if a metric has structures of both gradient shrinking Ricci soliton and half harmonic Weyl curvature, then except for three examples, it has to be an Einstein metric with positive scalar curvature. Precisely, we prove that a four-dimensional gradient shrinking Ricci soliton with $\delta W^{\pm} = 0$ is either Einstein, or a finite quotient of $S^3 \times \mathbb{R}$, $S^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$. We also prove that a four-dimensional gradient Ricci soliton with constant scalar curvature is either Kähler–Einstein, or a finite quotient of $M \times \mathbb{C}$, where M is a Riemann surface. The method of our proof is to construct a weighted subharmonic function using curvature decompositions and the Weitzenböck formula for half Weyl curvature, and the method was motivated by previous work (Gursky and LeBrun in Ann Glob Anal Geom 17:315–328, 1999; Wu in Einstein four-manifolds of three-nonnegative curvature operator 2013; Trans Am Math Soc 369:1079–1096, 2017; Yang in Invent Math 142:435–450, 2000) on the rigidity of Einstein four-manifolds with positive sectional curvature, and previous work (Cao and Chen in Trans Am Math Soc 364:2377–2391, 2012; Duke Math J 162:1003–1204, 2013; Catino in Math Ann 35:629–635, 2013) on the rigidity of gradient Ricci solitons.

Mathematics Subject Classification Primary 53C24 · 53C25

Communicated by J. Jost.

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1 Introduction

In this paper we investigate four-dimensional gradient shrinking Ricci solitons with half harmonic Weyl curvature ($\delta W^{\pm} = 0$). A Riemannian metric g on a smooth manifold M^n is called a gradient Ricci soliton, if there exist an $f \in C^{\infty}(M)$ and a $\lambda \in \mathbb{R}$, such that

$$\mathrm{Ric} + \nabla^2 f = \lambda g. \tag{1}$$

The function f is called a potential function for the gradient Ricci soliton. By convention we denote it by the triple (M^n, g, f). The gradient Ricci soliton is called shrinking, steady, or expanding, if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. A gradient Ricci soliton is a natural extension of an Einstein metric, since the soliton equation (1) becomes an Einstein metric equation when f is a constant function. Gradient Ricci solitons play an important role in Hamilton's Ricci flow [22–24] and Perelman's [34–36] resolutions of the Poincaré Conjecture and Thurston's Geometrization Conjecture, as they are self-similar solutions and possible singular models of the Ricci flow, and critical points of Perelman's entropies. See [5] for an excellent survey.

There has been lots of effort to understand the geometry of gradient Ricci solitons, especially their classifications. For our purpose we list previous results for gradient shrinking Ricci solitons.

For dimensions 2 and 3, following [8,23,25,32,35], the classification is complete.

For dimensions equal to or greater than 4. Following Ni and Wallach [32] and Zhang [46] (see also alternative proofs in [10,16,32,37,46]), a locally conformally flat ($W = 0$) gradient shrinking Ricci soliton is a finite quotient of S^n, $S^{n-1} \times \mathbb{R}$, or $\mathbb{R}^n$. Fernández-López and García-Río [18], and Munteanu and Sesum [29] proved a gradient shrinking Ricci soliton with harmonic Weyl curvature ($\delta W = 0$) is either Einstein, or a finite quotient of $N^k \times \mathbb{R}^{n-k}$ for $0 \leq k \leq n$, where N^k is a k-dimensional Einstein manifold of positive scalar curvature. Cao and Chen [7] proved that a Bach-flat gradient shrinking Ricci soliton is either Einstein, or a finite quotient of $N^{n-1} \times \mathbb{R}$ or $\mathbb{R}^n$, where N^{n-1} is an $(n-1)$-dimensional Einstein manifold. Various rigidity results under appropriate curvature pinching assumptions were proved in [2,11,30,31], etc.

In particular, in dimension 4, by the duality decomposition, it is natural to consider half curvature tensor. As is well-known, a Riemannian metric with $\delta W^{\pm} = 0$ is, among others, another interesting extension of an Einstein metric on a four-manifold [1], see for example Gursky [20] for an interesting gap theorem for $\|W^{\pm}\|_{L^2}$. Chen and Wang [14] (see also Cao and Chen [7]) proved that a half conformally flat ($W^{\pm} = 0$) four-dimensional gradient shrinking Ricci soliton is a finite quotient of S^4, $\mathbb{C}P^2$, $S^3 \times \mathbb{R}$, or $\mathbb{R}^4$. In [44], the second author observed that a compact four-dimensional gradient shrinking Ricci soliton with $\delta W^{\pm} = 0$ and half nonnegative isotropic curvature is a finite quotient of S^4 or a Kähler–Einstein four-manifold.

In this paper we classify four-dimensional gradient shrinking Ricci solitons with harmonic half Weyl curvature,

Theorem 1.1 *A four-dimensional gradient shrinking Ricci soliton with $\delta W^{\pm} = 0$ is either Einstein, or a finite quotient of $S^3 \times \mathbb{R}$, $S^2 \times \mathbb{R}^2$, or $\mathbb{R}^4$.*

Observe that on a four-manifold any Kähler metric with constant scalar curvature satisfies $\delta W^{+} = 0$ and $\frac{|W^{+}|^2}{R^2} = \frac{1}{24}$ automatically (see [15]), therefore we show

Theorem 1.2 *A four-dimensional gradient Kähler–Ricci soliton with constant scalar curvature is either Kähler–Einstein, or a finite quotient of $M \times \mathbb{C}$, where M is a Riemann surface.*

Recently, Fernández-López and García-Río [19] proved Theorem 1.2 using a different method, moreover they were able to classify six-dimensional gradient Kähler–Ricci solitons with constant scalar curvature.

Su and Zhang [40] proved that a complete noncompact gradient Kähler–Ricci soliton with vanishing Bochner tensor is Kähler–Einstein. Chen and Zhu [13] proved that a gradient steady Kähler–Ricci soliton with harmonic Bochner tensor is Calabi–Yau, and a gradient shrinking (expanding) Kähler–Ricci soliton with harmonic Bochner tensor is either Kähler–Einstein, or a finite quotient of $N^k \times \mathbb{C}^{n-k}$, where N^k is a Kähler–Einstein manifold of positive (negative) scalar curvature. Calamai and Petrecca [3] proved that an extremal Kähler–Ricci soliton with positive holomorphic sectional curvature is Kähler–Einstein.

Consequently we observe that the Kato inequality enables one to remove the nonnegative Ricci curvature assumption in Theorem 1.1 of Catino [11].

Theorem 1.3 (Catino [11]) *A gradient shrinking Ricci soliton with*

$$|W|R \leq \sqrt{\frac{2(n-1)}{n-2}}\left(\overset{\circ}{\mathrm{Ric}} - \frac{R}{\sqrt{n(n-1)}}\right)^2$$

is a finite quotient of S^n, $S^{n-1} \times \mathbb{R}$, or $\mathbb{R}^n$.

Catino's method was to prove that $\frac{|\mathrm{Ric}|^2}{R^2}$ is h-subharmonic ($h = f - 2\ln R$) under the Weyl curvature pinching condition, and he assumed $\mathrm{Ric} \geq 0$ to ensure that $\frac{|\mathrm{Ric}|^2}{R^2}$ is L_h^2-integrable, then he applied a weighted maximum principle. We observe that $\frac{|\mathrm{Ric}|}{R}$ is automatically L_h^2-integrable by a result of Munteanu and Sesum [29], and by the Kato inequality $|\nabla \mathrm{Ric}|^2 \geq |\nabla|\mathrm{Ric}||^2$, Catino's method shows that $\frac{|\mathrm{Ric}|}{R}$ is also h-subharmonic.

Remark 1.1 It is interesting to point out that the method of our proof is new, and is different from cases mentioned above. For example for gradient shrinking Ricci solitons with $\delta W = 0$, the proofs of Fernández-López and García-Río [18], and Munteanu and Sesum [29] rely on the following identity. If the Ricci curvature is bounded below and $|\mathrm{Rm}| \leq e^{a(r+1)}$ for some $a \in \mathbb{R}$ [9], or if $\int_M |\mathrm{Rm}|^2 e^{-\delta f} < \infty$ for some $\delta < 1$ [29], then

$$\int_M |\delta \mathrm{Rm}|^2 e^{-f} = \int_M |\nabla \mathrm{Ric}|^2 e^{-f}.$$

Unfortunately, it is not clear whether there is an analogue identity for half curvature tensor.

The method to prove Theorem 1.1 is motivated by previous work on the rigidity of Einstein four-manifolds by Gursky and LeBrun [21], Yang [45], and the second named author [43, 44], where the Weitzenböck formula plays a key role. The main ingredients are curvature decompostions and the Weitzenböck formula for half Weyl curvature [44]. First we show, using the curvature decomposition, that if $\delta W^{\pm} = 0$ then ∇f is an eigenvector of the Ricci tensor, and observe that $W^{\pm}$ can be expressed explicitly in terms of the traceless Ricci curvature. Next applying the Weitzenböck formula and the weighted maximum principle, we prove an identity involving the (anti-)self-dual Weyl curvature, the traceless Ricci curvature and the scalar curvature, which further implies that either $f \equiv$const, hence (M, g) is an Einstein manifold; or $W^{\pm} \equiv 0$, hence (M, g) is a finite quotient of $S^3 \times \mathbb{R}$ or $\mathbb{R}^4$; or $R \equiv$const and $0 \leq \mathrm{Ric} \leq \lambda g$, hence (M, g) is a finite quotient of $S^2 \times \mathbb{R}^2$.

The rest of the paper is organized as follows. In sect. 2, we discuss curvature decompositions and the relationship between $W^{\pm}$ and the traceless Ricci curvature when $\delta W^{\pm} = 0$. In sect. 3, we prove Theorems 1.1 and 1.2 by applying the Weitzenböck formula and the Yau–Naber maximum principle.

2 Curvature decompositions on four-dimensional gradient Ricci solitons

In this section we will discuss curvature decompositions on four-dimensional gradient Ricci solitons. First we fix some notation. Our sign conventions for the curvature tensor will be so that

$$R_{ijkl} = g_{hk} R^h_{ijl}, \quad K(e_i, e_j) = R_{ijij}, \quad R_{ik} = g^{jl} R_{ijkl}, \quad R = g^{ij} R_{ij}.$$

And our convention for the inner product of two (0, 4)-tensors S, T will be

$$\langle S, T\rangle = \frac{1}{4} S_{ijkl} T^{ijkl}$$

so that our convention agrees with the one in Derdzinski's Weitzenböck formula [15].

On an oriented 4-manifold, the Hodge star $\star : \wedge^2 M \to \wedge^2 M$ has eigenvalues 1 and -1. Thus we can break $\wedge^2 M = \wedge^+ M + \wedge^- M$ according to the eigenspaces of $\star$. Given a basis $\{e_1, e_2, e_3, e_4\}$ of $T_p M$, for any pair (ij), $1 \le i \ne j \le 4$, denote $(i'j')$ to be the dual of (ij), i.e., the pair such that $e_i \wedge e_j \pm e_{i'} \wedge e_{j'} \in \wedge^{\pm} M$. In other words, $(iji'j') = \sigma(1234)$ for some even permutation $\sigma \in S_4$. So for any (0, 4)-tensor T, its (anti-)self-dual part is

$$T^{\pm}_{ijkl} = \frac{1}{4}(T_{ijkl} \pm T_{ijk'l'} \pm T_{i'j'kl} + T_{i'j'k'l'}),$$

It is well known that for four-manifolds, Weyl curvature has a very interesting symmetry,

Lemma 2.1 *Let (M, g) be a four-dimensional Riemannian manifold. Then*

$$W_{ijkl} = W_{i'j'k'l'},$$

therefore,

$$W^{\pm}_{ijkl} = \pm\, W^{\pm}_{ijk'l'} = \pm W^{\pm}_{i'j'kl} = W^{\pm}_{i'j'k'l'} = \frac{1}{2}(W_{ijkl} \pm W_{ijk'l'}).$$

In particular, for any $u \in C^\infty(M)$,

$$|\iota_{\nabla u} W^{\pm}|^2 = \frac{1}{4} |W^{\pm}|^2 |\nabla u|^2.$$

Now we discuss curvature decompositions on four-dimensional gradient Ricci solitons. First we recall some basic identities,

Lemma 2.2 *Let (M, g, f) be a gradient Ricci soliton. Then*

$$\begin{aligned}
\nabla_k R_{jl} - \nabla_l R_{jk} &= R_{ijkl} \nabla^i f, \\
(\delta \mathrm{Rm})_{jkl} = \nabla^i R_{ijkl} &= R_{ijkl} \nabla^i f, \\
\nabla_i R = 2\nabla^j R_{ij} &= 2 R_{ij} \nabla^j f.
\end{aligned}$$

In [6,7], Cao and Chen introduced a (0, 3)-tensor $D = \frac{n-2}{n-3} \delta W - \iota_{\nabla f} W$, which plays an important role in their classification of locally conformally flat gradient steady Ricci solitons and Bach-flat gradient shrinking Ricci solitons. We observe that D and its "self-dual" and "anti-self-dual" parts $D^{\pm}$, arise naturally from the standard curvature decomposition. For our purpose we only calculate the four-dimensional case, for general dimensions the argument is the same.

Lemma 2.3 *Let (M, g, f) be a four-dimensional gradient Ricci soliton. Then*

$$
\begin{aligned}
D_{jkl} &= 2\nabla^i W_{ijkl} - W_{ijkl}\nabla^i f\\
&= \frac{1}{2}(R_{jl}\nabla_k f - R_{jk}\nabla_l f) + \frac{1}{12}(\nabla_k R g_{jl} - \nabla_l R g_{jk}) - \frac{R}{6}(g_{jl}\nabla_k f - g_{jk}\nabla_l f),\\
D^{\pm}_{jkl} &\triangleq 2\nabla^i W^{\pm}_{ijkl} - W^{\pm}_{ijkl}\nabla^i f\\
&= \frac{1}{4}(R_{jl}\nabla_k f - R_{jk}\nabla_l f) + \frac{1}{24}(\nabla_k R g_{jl} - \nabla_l R g_{jk})\\
&\quad - \frac{R}{12}(g_{jl}\nabla_k f - g_{jk}\nabla_l f)\\
&\quad \pm \frac{1}{4}(R_{jl'}\nabla_{k'} f - R_{jk'}\nabla_{l'} f) \pm \frac{1}{24}(\nabla_{k'} R g_{jl'} - \nabla_{l'} R g_{jk'})\\
&\quad \mp \frac{R}{12}(g_{jl'}\nabla_{k'} f - g_{jk'}\nabla_{l'} f).
\end{aligned}
$$

In particular,

$$
|D^+|^2 = |D^-|^2 = \frac{1}{2}|D|^2 = \frac{1}{4}|\overset{\circ}{\mathrm{Rc}}|^2|\nabla f|^2 - \frac{1}{48}|R\nabla f - 2\nabla R|^2. \tag{2}
$$

Proof We apply the standard curvature decomposition to both sides of the identity $\nabla^i R_{ijkl} = R_{ijkl}\nabla^i f$,

For the left hand side, we have

$$
\begin{aligned}
\nabla^i R_{ijkl} &= \nabla^i W_{ijkl} + \frac{1}{2}\nabla^i(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il})\\
&\quad - \frac{\nabla^i R}{6}(g_{ik}g_{jl} - g_{il}g_{jk})\\
&= \nabla^i W_{ijkl} + \frac{1}{2}(\nabla_k R_{jl} - \nabla_l R_{jk} + \frac{1}{2}\nabla_k R g_{jl} - \frac{1}{2}\nabla_l R g_{jk})\\
&\quad - \frac{1}{6}(\nabla_k R g_{jl} - \nabla_l R g_{jk})\\
&= \nabla^i W_{ijkl} + \frac{1}{2}\nabla^i R_{ijkl} + \frac{1}{4}(\nabla_k R g_{jl} - \nabla_l R g_{jk}) - \frac{1}{6}(\nabla_k R g_{jl} - \nabla_l R g_{jk}),
\end{aligned}
$$

Therefore we get

$$
\nabla^i R_{ijkl} = 2\nabla^i W_{ijkl} + \frac{1}{6}(\nabla_k R g_{jl} - \nabla_l R g_{jk}),
$$

On the other hand, observe that by the second Bianchi identity, we have $\nabla^i R_{i'j'kl} = 0$, so we get

$$
\begin{aligned}
4\nabla^i R^{\pm}_{ijkl} &= \nabla^i(R_{ijkl} + R_{ijk'l'} + R_{i'j'kl} + R_{i'j'k'l'})\\
&= \nabla^i(R_{ijkl} + R_{ijk'l'})\\
&= 2\nabla^i(W_{ijkl} + W_{ijk'l'}) + \frac{1}{6}(\nabla_k R g_{jl} - \nabla_l R g_{jk}) + \frac{1}{6}(\nabla_{k'} R g_{jl'} - \nabla_{l'} R g_{jk'})\\
&= 4\nabla^i W^{\pm}_{ijkl} + \frac{1}{6}(\nabla_k R g_{jl} - \nabla_l R g_{jk}) + \frac{1}{6}(\nabla_{k'} R g_{jl'} - \nabla_{l'} R g_{jk'}),
\end{aligned}
$$

therefore we obtain

$$
\begin{aligned}
R_{ijkl}\nabla^i f + R_{ijk'l'}\nabla^i f = 4\nabla^i W^{\pm}_{ijkl} + \frac{1}{6}(\nabla_k R g_{jl} - \nabla_l R g_{jk}) \\
+ rac16(\nabla_{k'} R g_{jl'} - \nabla_{l'} R g_{jk'})
\end{aligned} \tag{3}
$$

For the right hand side, we have

$$
\begin{aligned}
R_{ijkl}\nabla^i f &= W_{ijkl}\nabla^i f + \frac{1}{2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il})\nabla^i f \\
&\quad - \frac{R}{6}(g_{ik}g_{jl} - g_{il}g_{jk})\nabla^i f \\
&= W_{ijkl}\nabla^i f + \frac{1}{2}(R_{jl}\nabla_k f - R_{jk}\nabla_l f + \frac{1}{2}\nabla_k R g_{jl} - \frac{1}{2}\nabla_l R g_{jk}) \\
&\quad - \frac{R}{6}(g_{jl}\nabla_k f - g_{jk}\nabla_l f).
\end{aligned}
$$

Taking the difference, we get

$$
\begin{aligned}
D_{jkl} &= 2\nabla^i W_{ijkl} - W_{ijkl}\nabla^i f \\
&= \frac{1}{2}(R_{jl}\nabla_k f - R_{jk}\nabla_l f) + \frac{1}{12}(\nabla_k R g_{jl} - \nabla_l R g_{jk}) - \frac{R}{6}(g_{jl}\nabla_k f - g_{jk}\nabla_l f),
\end{aligned}
$$

Therefore

$$
\begin{aligned}
D^{\pm}_{jkl} &= 2\nabla^i W^{\pm}_{ijkl} - W^{\pm}_{ijkl}\nabla^i f \\
&= \frac{1}{2}[(2\nabla^i W_{ijkl} - W_{ijkl}\nabla^i f) \pm (2\nabla^i W_{ijk'l'} - W_{ijk'l'}\nabla^i f)] \\
&= \frac{1}{4}(R_{jl}\nabla_k f - R_{jk}\nabla_l f) + \frac{1}{24}(\nabla_k R g_{jl} - \nabla_l R g_{jk}) - \frac{R}{12}(g_{jl}\nabla_k f - g_{jk}\nabla_l f) \\
&\quad \pm \frac{1}{4}(R_{jl'}\nabla_{k'} f - R_{jk'}\nabla_{l'} f) \pm \frac{1}{24}(\nabla_{k'} R g_{jl'} - \nabla_{l'} R g_{jk'}) \\
&\quad \mp \frac{R}{12}(g_{jl'}\nabla_{k'} f - g_{jk'}\nabla_{l'} f),
\end{aligned}
$$

□

Fernández-López and García-Río [18] proved that if a gradient Ricci soliton satisfies $\delta W = 0$, then ∇f is an eigenvector of the Ricci tensor. Following from Lemma 2.3, it is easy to see that in dimension four, $\delta W^{\pm} = 0$ provides the same information,

Lemma 2.4 *Let (M, g, f) be a four-dimensional gradient Ricci soliton. If $\delta W^{\pm} = 0$, then ∇f, whenever nonzero, is an eigenvector of the Ricci tensor.*

Proof In Eq. (3), if $\delta W^{\pm} = 0$, then

$$
R_{ijkl}\nabla^i f + R_{ijk'l'}\nabla^i f = \frac{1}{6}(\nabla_k R g_{jl} - \nabla_l R g_{jk}) + \frac{1}{6}(\nabla_{k'} R g_{jl'} - \nabla_{l'} R g_{jk'})
$$

Let $e_1 = \frac{\nabla f}{|\nabla f|}$, and extend it to an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T_p M$. Let $j = k = 1, l \neq 1$, then since $(klk'l') = \sigma(1234)$, we have $g_{jl} = g_{jk'} = g_{jl'} = 0$. Therefore we get

$$
0 = \nabla_l R = 2R_{lj}\nabla^j f = 2|\nabla f| R_{1l}.
$$

□

Combining Lemmas 2.3 and 2.4, we make a key observation that if $\delta W^{\pm}=0$, then $W^{\pm}$ has a nice expression in terms of Ricci curvature and scalar curvature.

Proposition 2.1 *Let (M, g, f) be a four-dimensional gradient Ricci soliton with $\delta W^{\pm}=0$. Denote a_1, a_2, a_3, a_4 be the eigenvalues of the traceless Ricci tensor with corresponding eigenvectors $e_1=\frac{\nabla f}{|\nabla f|}, e_2, e_3, e_4$. Then whenever $\nabla f\neq 0$,*

$$
\begin{aligned}
b_1 &\triangleq W^{\pm}_{1212}=-\frac{1}{12}(a_1+3a_2)=\frac{1}{12}(a_3+a_4-2a_2),\\
b_2 &\triangleq W^{\pm}_{1313}=-\frac{1}{12}(a_1+3a_3)=\frac{1}{12}(a_2+a_4-2a_3),\\
b_3 &\triangleq W^{\pm}_{1414}=-\frac{1}{12}(a_1+3a_4)=\frac{1}{12}(a_2+a_3-2a_4),\\
& W^{\pm}_{1j1l}=0,\quad \text{if } j\neq l.
\end{aligned}
$$

Proof By Lemma 2.4 we have $R_{1j}=0$ for $j\neq 1$, which gives us

$$\nabla_1 R=2R_{1j}\nabla^j f=2R_{11}|\nabla f|.$$

If $\delta W^{\pm}=0$, then

$$
\begin{aligned}
-W^{\pm}_{1j1j}|\nabla f| &= -W^{\pm}_{ij1j}\nabla^i f\\
&=\frac{1}{4}(R_{jj}\nabla_1 f-R_{1j}\nabla_j f)+\frac{1}{24}(\nabla_1 R g_{jj}-\nabla_j R g_{1j})\\
&\quad-\frac{R}{12}(g_{jj}\nabla_1 f-g_{1j}\nabla_j f)\\
&\quad\pm\frac{1}{4}(R_{jj'}\nabla_{1'} f-R_{j1'}\nabla_{j'} f)\pm\frac{1}{24}(\nabla_{1'} R g_{jj'}-\nabla_{j'} R g_{j1'})\\
&\quad\mp\frac{R}{12}(g_{jj'}\nabla_{1'} f-g_{j1'}\nabla_{j'} f)\\
&=\frac{1}{4}R_{jj}|\nabla f|+\frac{1}{12}R_{11}|\nabla f|-\frac{R}{12}|\nabla f|.
\end{aligned}
$$

If $\nabla f\neq 0$, then we get

$$
\begin{aligned}
-W^{\pm}_{1212} &= \frac{1}{4}R_{22}+\frac{1}{12}R_{11}-\frac{R}{12}\\
&=\frac{1}{12}\left[3\left(R_{22}-\frac{R}{4}\right)+\left(R_{11}-\frac{R}{4}\right)\right]\\
&=\frac{1}{12}(a_1+3a_2),
\end{aligned}
$$

similarly we get $W^{\pm}_{1313}$ and $W^{\pm}_{1414}$.

If $j\neq l$, then it is easy to compute that $W^{\pm}_{1j1l}=0$. □

3 Proof of Theorem 1.1

First recall the Weitzenböck formula for the half Weyl curvature $W^{\pm}$, which was proved by the second named author in [44],

Proposition 3.1 ([44]) *Let (M, g, f) be a four-dimensional gradient Ricci soliton. Then*

$$\Delta_f|W^\pm|^2 = 2|\nabla W^\pm|^2 + 4\lambda|W^\pm|^2 - 36\det W^\pm - \langle(\mathring{\mathrm{Ric}}\circ\mathring{\mathrm{Ric}})^\pm, W^\pm\rangle.$$

Next we compute,

Proposition 3.2 *Let (M, g, f) be a four-dimensional gradient Ricci soliton. Let $h = f - \ln R^2$, then*

$$\Delta_h\left(\frac{|W^\pm|}{R}\right) \geq \frac{1}{2|W^\pm|R^2}\Big(R^2|W^\pm|^2 - 36R\det W^\pm + 4|W^\pm|^2|\mathring{\mathrm{Ric}}|^2 - R\langle(\mathring{\mathrm{Ric}}\circ\mathring{\mathrm{Ric}})^\pm, W^\pm\rangle\Big). \tag{4}$$

Proof Recall the Kato inequality $|\nabla T|^2 \geq |\nabla|T||^2$ for any tensor T. From Proposition 3.1, we get

$$\begin{aligned}\Delta_f|W^\pm| &= \frac{1}{2|W^\pm|}\Big[2|\nabla W^\pm|^2 - 2|\nabla|W^\pm||^2 + 4\lambda|W^\pm|^2 - 36\det W^\pm \\ &\quad - \langle(\mathring{\mathrm{Ric}}\circ\mathring{\mathrm{Ric}})^\pm, W^\pm\rangle\Big] \\ &\geq \frac{1}{2|W^\pm|}\Big[4\lambda|W^\pm|^2 - 36\det W^\pm - \langle(\mathring{\mathrm{Ric}}\circ\mathring{\mathrm{Ric}})^\pm, W^\pm\rangle\Big]\end{aligned}$$

and recall that

$$\Delta_f R = 2\lambda R - 2|\mathrm{Ric}|^2. \tag{5}$$

Therefore we compute

$$\begin{aligned}\Delta_f\left(\frac{|W^\pm|}{R}\right) &= \frac{\Delta_f|W^\pm|}{R} - \frac{|W^\pm|\Delta_f R}{R^2} - 2\frac{\nabla|W^\pm|\nabla R}{R^2} + 2\frac{|W^\pm||\nabla R|^2}{R^3} \\ &\geq \frac{1}{2R|W^\pm|}(4\lambda|W^\pm|^2 - 36\det W^\pm - \langle(\mathring{\mathrm{Ric}}\circ\mathring{\mathrm{Ric}})^\pm, W^\pm\rangle) \\ &\quad - \frac{|W^\pm|}{R^2}(2\lambda R - 2|\mathrm{Ric}|^2) \\ &\quad - 2\frac{\nabla|W^\pm|\nabla R}{R^2} + 2\frac{|W^\pm||\nabla R|^2}{R^3} \\ &= -2\frac{1}{R}\Big\langle\nabla\Big(\frac{|W^\pm|}{R}\Big), \nabla R\Big\rangle + \frac{1}{2|W^\pm|R^2}\Big(4|W^\pm|^2|\mathrm{Ric}|^2 \\ &\quad - 36R\det W^\pm - R\langle(\mathring{\mathrm{Ric}}\circ\mathring{\mathrm{Ric}})^\pm, W^\pm\rangle\Big) \\ &= -\Big\langle\nabla\Big(\frac{|W^\pm|}{R}\Big), \nabla\ln R^2\Big\rangle + \frac{1}{2|W^\pm|R^2}\Big(R^2|W^\pm|^2 \\ &\quad - 36R\det W^\pm + 4|W^\pm|^2|\mathring{\mathrm{Ric}}|^2 - R\langle(\mathring{\mathrm{Ric}}\circ\mathring{\mathrm{Ric}})^\pm, W^\pm\rangle\Big).\end{aligned}$$

□

We have,

Lemma 3.1 *Let (M, g, f) be a four-dimensional gradient shrinking Ricci soliton with $\delta W^\pm = 0$, then whenever $\nabla f \neq 0$,*

$$R^2|W^\pm|^2 - 36R\det W^\pm + 4|W^\pm|^2|\mathring{\mathrm{Ric}}|^2 - R\langle(\mathring{\mathrm{Ric}}\circ\mathring{\mathrm{Ric}})^\pm, W^\pm\rangle \geq 0,$$

Denote a_1, a_2, a_3, a_4 *be eigenvalues of* $\overset{\circ}{\mathrm{Ric}}$ *with corresponding orthonormal eigenvectors* $e_1 = \frac{\nabla f}{|\nabla f|}, e_2, e_3, e_4$. *The equality holds if and only if, either*

(1). $a_2 = a_3 = a_4$, *i.e.,* $W^{\pm} = 0$; *or*

(2). *after a rearrangement of* $\{e_2, e_3, e_4\}$, $a_1 = a_2 = -a, a_3 = a_4 = a$, *and* $R = 4a$ *for some* $a > 0$.

Remark 3.1 Lemma 3.1 also works for gradient steady and expanding solitons, and the sign of a in the second equality case changes correspondingly.

The proof of Lemma 3.1 will be presented at the end of this section. We first prove Theorem 1.1.

Proof of Theorem 1.1. Chen [12] proved that any gradient shrinking Ricci soliton has $R \geq 0$. Moreover, either $R > 0$ on M, or $R \equiv 0$ on M, and if $R \equiv 0$ then (M, g) is a finite quotient of $\mathbb{R}^4$, see [38,39]. From now on we assume $R > 0$.

If M is compact, then from Proposition 3.2,

$$\begin{aligned} 0 &= \int_M \Delta_h \left(\frac{|W^{\pm}|}{R}\right) e^{-h} dv \\ &\geq \int_M \frac{1}{2|W^{\pm}|} \Big(R^2|W^{\pm}|^2 - 36R \det W^{\pm} + 4|W^{\pm}|^2|\overset{\circ}{\mathrm{Ric}}|^2 \\ &\quad - R\langle(\overset{\circ}{\mathrm{Ric}} \circ \overset{\circ}{\mathrm{Ric}})^{\pm}, W^{\pm}\rangle\Big) e^{-f} dv \end{aligned}$$

so by Lemma 3.1 we get

$$R^2|W^{\pm}|^2 - 36R \det W^{\pm} + 4|W^{\pm}|^2|\overset{\circ}{\mathrm{Ric}}|^2 - R\langle(\overset{\circ}{\mathrm{Ric}} \circ \overset{\circ}{\mathrm{Ric}})^{\pm}, W^{\pm}\rangle \equiv 0. \tag{6}$$

If M is noncompact, by Eq. (2) in Lemma 2.3 and the last equality in Lemma 2.1, if $\delta W^{\pm} = 0$, then

$$|W^{\pm}| \leq |\overset{\circ}{\mathrm{Ric}}| < |\mathrm{Ric}|,$$

Munteanu and Sesum [29] proved that for a gradient shrinking Ricci soliton,

$$\int_M |\mathrm{Ric}|^2 e^{-\delta f} dv < \infty,$$

for any $\delta > 0$. Therefore if $\delta W^{\pm} = 0$, then

$$\left\| \frac{|W^{\pm}|}{R} \right\|_{L^2_h(M)} = \int_M |W^{\pm}|^2 e^{-f} dv < \infty.$$

By a maximum principle of Naber [31] and Petersen and Wylie [37], if $\int_M e^{-h} dv < \infty$, then any L^2_h-integrable h-subharmonic function is constant, therefore we conclude that $\frac{|W^{\pm}|}{R}$ is constant, which also implies Eq. (6).

Recall that any gradient Ricci soliton is a real-analytic manifold (see [26] or [28]), hence all $|\nabla f|^2$, $|W^{\pm}|^2$, and R are analytic functions on M, therefore either $\nabla f \equiv 0$ or $W^{\pm} \equiv 0$, or the second equality case in Lemma 3.1 holds on M.

Case 1. If $\nabla f \equiv 0$ on M,then (M, g) is Einstein.

Case 2. If $W^{\pm} \equiv 0$ on M, then (M, g) is a finite quotient of $S^3 \times \mathbb{R}$ or $\mathbb{R}^4$ by [14].

Case 3. If $\nabla f \not\equiv 0$ and $W^{\pm} \not\equiv 0$, then the second case in Lemma 3.1 holds in an open dense set S of M. Without loss of generality, assume $\delta W^{+} = 0$. First it is easy to see that

$$R_{11} = \overset{\circ}{R}_{11} + \frac{R}{4} = 0, \quad R_{22} = 0, \quad R_{33} = R_{44} = 2a,$$

therefore by Lemma 2.4, $\nabla R = \nabla_1 R e_1 = 2R_{11}|\nabla f| e_1 = 0$, that is $R \equiv$const on S. By the continuity of R, we have $R \equiv$const on M. Furthermore, since the eigenvalues of Ricci curvature are $0, 0, 2a, 2a$, so we have $|\mathrm{Ric}|^2 = 8a^2$, plugging into Eq. (5), we get

$$\begin{aligned} 0 = \Delta_f R &= 2\lambda R - 2|\mathrm{Ric}|^2 \\ &= 8\lambda a - 16a^2, \end{aligned}$$

therefore $a = \frac{\lambda}{2}$, which in particular implies that $0 \le \mathrm{Ric} \le \lambda g$. By Proposition 1.3 in [38], (M, g, f) is rigid, i.e., it is a finite quotient of $N^k \times \mathbb{R}^{4-k}$, where N^k is an Einstein manifold. Since (M, g, f) is neither Einstein nor half conformally flat, we conclude that (M, g) is a finite quotient of $S^2 \times \mathbb{R}^2$. □

Proof of Theorem 1.2. On a four-manifold, a Kähler metric with constant scalar curvature satisfies

$$\delta W^{+} = 0, \quad \frac{|W^{+}|^2}{R^2} = \frac{1}{24}.$$

If (M, g, f) is a gradient shrinking Ricci soliton, it follows directly from Case 3 of the proof of Theorem 1.1.

If (M, g, f) is a gradient expanding Ricci soliton, then by Proposition 3.2 and Lemma 3.1 we have

$$R^2|W^{+}|^2 - 36R \det W^{+} + 4|W^{+}|^2|\overset{\circ}{\mathrm{Ric}}|^2 - R\langle(\overset{\circ}{\mathrm{Ric}} \circ \overset{\circ}{\mathrm{Ric}})^{+}, W^{+}\rangle \equiv 0.$$

Similar to the proof of Theorem 1.1, there are three cases,

Case 1. If $\nabla f \equiv 0$ on M, then (M, g) is Kähler–Einstein.

Case 2. If $W^{+} \equiv 0$ on M, then $D^{+} \equiv 0$, hence by Lemma 2.3, $D \equiv 0$, therefore $W \equiv 0$ by Theorem 5.1 in Cao and Chen [7], and (M, g, f) is a finite quotient of Gaussian expanding soliton by Su and Zhang [40].

Case 3. If $\nabla f \not\equiv 0$ and $W^{+} \not\equiv 0$, then it follows from Case 3 in the proof of Theorem 1.1 that it is rigid, hence a finite quotient of $M \times \mathbb{C}$, where M is a Riemann surface of constant negative curvature.

□

Proof of Lemma 3.1. By proposition 2.1, we express each term in terms of eigenvalues of traceless Ricci tensor,

$$
\begin{aligned}
|\mathring{\mathrm{Rc}}|^2 &= a_1^2+a_2^2+a_3^2+a_4^2\\
&= 2\big(a_2^2+a_3^2+a_4^2+a_2a_3+a_2a_4+a_3a_4\big),\\
|W^{\pm}|^2 &= 4(b_1^2+b_2^2+b_3^2)\\
&= \frac{1}{6}\big(a_2^2+a_3^2+a_4^2-a_2a_3-a_2a_4-a_3a_4\big)\\
36\det W^{\pm} &= \frac{1}{6}\big(a_3+a_4-2a_2\big)\big(a_2+a_4-2a_3\big)\big(a_2+a_3-2a_4\big)\\
&= \frac{1}{6}\big(-2a_2^3-2a_3^3-2a_4^3+3a_2^2a_3+3a_3^2a_2+3a_2^2a_4\\
&\quad +3a_4^2a_2+3a_3^2a_4+3a_4^2a_3-12a_2a_3a_4\big),\\
\langle(\mathring{\mathrm{Rc}}\circ\mathring{\mathrm{Rc}})^{\pm},W^{\pm}\rangle &= W^{\pm}_{ijij}\mathring{R}_{ii}\mathring{R}_{jj}\\
&= 2\left[b_1\big(a_1a_2+a_3a_4\big)+b_2\big(a_1a_3+a_2a_4\big)+b_3\big(a_1a_4+a_2a_3\big)\right]\\
&= \frac{1}{6}\big(2a_2^3+2a_3^3+2a_4^3+a_2^2a_3+a_3^2a_2+a_2^2a_4\\
&\quad +a_4^2a_2+a_3^2a_4+a_4^2a_3-12a_2a_3a_4\big).
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
6\phi &= 6(R^2|W^{\pm}|^2-36R\det W^{\pm}+4|W^{\pm}|^2|\mathring{\mathrm{Ric}}|^2-R\langle(\mathring{\mathrm{Ric}}\circ\mathring{\mathrm{Ric}})^{\pm},W^{\pm}\rangle)\\
&= R^2\big(a_2^2+a_3^2+a_4^2-a_2a_3-a_2a_4-a_3a_4\big)\\
&\quad -4R\big(a_2^2a_3+a_3^2a_2+a_2^2a_4+a_4^2a_2+a_3^2a_4+a_4^2a_3-6a_2a_3a_4\big)\\
&\quad +8\big(a_2^2+a_3^2+a_4^2+a_2a_3+a_2a_4+a_3a_4\big)\times\\
&\quad \big(a_2^2+a_3^2+a_4^2-a_2a_3-a_2a_4-a_3a_4\big).
\end{aligned}
\tag{7}
$$

By abusing the noataion, we identify ϕ and 6ϕ. Observe that ϕ is a fourth-order homogeneous symmetric polynomial of a_2, a_3, a_4 if we rewrite $R = k(a_2+a_3+a_4)$ for some $k \in \mathbb{R}$.

First we show $\phi(a_2, a_3, a_4) \geq 0$ using Timofte's criterion for positivity of homogeneous symmetric polynomials (see Corollary 5.6 in [41]),

Proposition 3.3 (Timofte [41]) *Let p be a fourth-order homogeneous symmetric polynomial on $\mathbb{R}^n$, then*

$$
p \geq 0 \text{ on } \mathbb{R}^n \quad\Longleftrightarrow\quad p(t\cdot\mathbf{1}_{\mathbb{R}^i},\mathbf{1}_{\mathbb{R}^{n-i}}) \geq 0,\ \forall\, t\in[-1,1],\ i=1,2,\dots,n-1,
$$

where $\mathbf{1} = (1, 1, \dots, 1)$.

If $a_1 \neq 0$, without of loss of generality, assume $R = -ka_1 = k(a_2+a_3+a_4)$. In our case $n = 3$, so we need to show that

$$
\phi(t,1,1)\geq 0, \quad \phi(t,t,1)\geq 0, \quad \forall\, t\in[-1,1].
$$

For $\phi(t, 1, 1)$, plugging into equation (7), recall that $R = k(t + 2)$, we get

$$
\phi(t,1,1) = (t-1)^2\Big[k^2(t+2)^2-8k(t+2)+8(t^2+2t+3)\Big].
$$

Consider $\phi(t,1,1)$ as a quadratic function of k. When $-1 \le t \le 1$, the discriminant

$$\mathfrak{D} = -32(t+2)^2(t-1)^4(t+1)^2 \le 0,$$

and $\mathfrak{D} < 0$ when $-1 < t < 1$. Therefore for all $-1 \le t \le 1$,

$$\phi(t,1,1) \ge 0,$$

and $\phi(t,1,1) = 0$ if and only if $t = 1$, or $t = -1$ and $R = 4$.

For $\phi(t,t,1)$, since we assume $a_1 \ne 0$, so $t \ne -\frac{1}{2}$. Plugging into Eq. (7), recall that $R = k(2t+1)$, we get

$$\phi(t,t,1) = (t-1)^2\Big[k^2(2t+1)^2 - 8kt(2t+1) + 8(3t^2+2t+1)\Big].$$

Consider $\phi(t,t,1)$ as a quadratic function of k, we see that when $-1 \le t \le 1$ and $t \ne -\frac{1}{2}$, the discriminant

$$\mathfrak{D} = -32(2t+1)^2(t-1)^4(t+1)^2 \le 0,$$

and $\mathfrak{D} < 0$ when $-1 < t < 1$ and $t \ne -\frac{1}{2}$. Therefore for all $-1 \le t \le 1$ and $t \ne -\frac{1}{2}$,

$$\phi(t,t,1) \ge 0,$$

and $\phi(t,t,1) = 0$ if and only if $t = 1$, or $t = -1$ and $R = -4$.

If $a_1 = 0$ (which corresponds to $t = -\frac{1}{2}$ in $\phi(t,t,1)$), i.e., $a_2 + a_3 + a_4 = 0$, then ϕ can be simplified as

$$\phi = 3R^2\big(a_2^2 + a_3^2 + a_2a_3\big) - 36Ra_2a_3(a_2+a_3) + 24\big(a_2^2 + a_3^2 + a_2a_3\big)^2.$$

Consider ϕ as a quadratic function of R, then its discriminant

$$\mathfrak{D} = 36^2a_2^2a_3^2(a_2+a_3)^2 - 36\left[a_2^2 + a_3^2 + (a_2+a_3)^2\right]^3 \le 0.$$

Recall a well-known inequality: if $a+b+c = 0$, then $3\sqrt{6}|abc| \le (a^2+b^2+c^2)^{\frac{3}{2}}$, with equality if and only if $a = -2b$, or $b = -2c$, or $c = -2a$. Since $-a_2 - a_3 + (a_2+a_3) = 0$, we have

$$\begin{aligned}
\mathfrak{D} &= 36^2a_2^2a_3^2(a_2+a_3)^2 - 36\left[a_2^2 + a_3^2 + (a_2+a_3)^2\right]^3 \\
&\le 24\left[a_2^2 + a_3^2 + (a_2+a_3)^2\right]^3 - 36\left[a_2^2 + a_3^2 + (a_2+a_3)^2\right]^3 \\
&\le 0,
\end{aligned}$$

with equality if and only if $a_2 = a_3 = 0$. So $\phi \ge 0$, and $\phi = 0$ if and only if $a_2 = a_3 = a_4 = 0$.

Therefore we proved that $\phi(R, a_2, a_3, a_4) \ge 0$ on $\mathbb{R}^4$.

Next we show that when $a_2 \ne a_3 \ne a_4$, then $\phi > 0$.

Assume that $a_2 \ne a_3 \ne a_4$ and $\phi(R, a_2, a_3, a_4) = 0$. Taking the first derivatives we get,

$$\begin{aligned}
\phi_{a_2} = {}& R^2(2a_2 - a_3 - a_4) - 4R\Big(2a_2a_3 + 2a_2a_4 + a_3^2 + a_4^2 - 6a_3a_4\Big) \\
&+ 16\Big(2a_2^3 + a_2a_3^2 + a_2a_4^2 - a_3^2a_4 - a_3a_4^2 - 2a_2a_3a_4\Big) = 0, \\
\phi_{a_3} = {}& R^2(2a_3 - a_2 - a_4) - 4R\Big(2a_2a_3 + 2a_3a_4 + a_2^2 + a_4^2 - 6a_2a_4\Big) \\
&+ 16\Big(2a_3^3 + a_2^2a_3 + a_3a_4^2 - a_2^2a_4 - a_2a_4^2 - 2a_2a_3a_4\Big) = 0, \\
\phi_{a_4} = {}& R^2(2a_4 - a_2 - a_2) - 4R\Big(2a_2a_4 + 2a_3a_4 + a_2^2 + a_3^2 - 6a_2a_3\Big) \\
&+ 16\Big(2a_4^3 + a_2^2a_4 + a_3^2a_4 - a_2^2a_3 - a_2a_3^2 - 2a_2a_3a_4\Big) = 0.
\end{aligned}$$

Taking the difference, since $a_2 \neq a_3 \neq a_4$, we get

$$\begin{aligned}
0 = \frac{\phi_{a_2} - \phi_{a_3}}{a_2 - a_3} =& 3R^2 - R(4(a_2 + a_3) - 32a_4) \\
&+ 16\Big(2a_2^2 + 2a_3^2 + 2a_4^2 + a_2a_3 + a_2a_4 + a_3a_4\Big), \\
0 = \frac{\phi_{a_2} - \phi_{a_4}}{a_2 - a_4} =& 3R^2 - R(4(a_2 + a_4) - 32a_3) \\
&+ 16\Big(2a_2^2 + 2a_3^2 + 2a_4^2 + a_2a_3 + a_2a_4 + a_3a_4\Big), \\
0 = \frac{\phi_{a_3} - \phi_{a_4}}{a_3 - a_4} =& 3R^2 - R(4(a_3 + a_4) - 32a_2) \\
&+ 16\Big(2a_2^2 + 2a_3^2 + 2a_4^2 + a_2a_3 + a_2a_4 + a_3a_4\Big).
\end{aligned}$$

Taking the difference again, we get $a_2 = a_3 = a_4$, contradiction! So $\phi > 0$ when $a_2 \neq a_3 \neq a_4$.

Therefore $\phi \geq 0$, and by Timofte's criterion and above argument, $\phi = 0$ if and only if, either $a_2 = a_3 = a_4$, i.e. $W^{\pm} = 0$;
or $a_1 = a_i = -a$, $a_j = a_k = a$, $2 \leq i \neq j \neq k \leq 4$, and $R = 4a$, for some $a > 0$.

□

Acknowledgements Part of the work was done when the first author was visiting the Department of Mathematics at Cornell University, he greatly thanks Professor Xiaodong Cao for his help and Department of Mathematics for their hospitality. The second author thanks Professors Jeffrey Case, Yuanqi Wang and Yuan Yuan for helpful discussions. We thank the anonymous referee for many valuable suggestions. The first author was partially supported by the China Scholarship Council (201208310431), NSFC (11671141) and the Natural Science Foundation of Shanghai (17ZR1412800). The second author was partially supported by an AMS-Simons postdoc travel grant, China Recruit Program for Global Young Talents, and NSFC (11701093). The third author was partially supported by a grant from the Simons Foundation (355608, William Wylie) and the NSF (1654034).

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