

SOME CURVATURE PINCHING RESULTS FOR RIEMANNIAN MANIFOLDS WITH DENSITY

WILLIAM WYLIE

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ABSTRACT. In this note we consider versions of both Ricci and sectional curvature pinching for Riemannian manifolds with density. In the Ricci curvature case the main result implies a diameter estimate that is new even for compact shrinking Ricci solitons. In the case of sectional curvature we prove a new sphere theorem.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold and X be a smooth vector field on M . There are a number of different “weighted” curvatures defined for the triple (M, g, X) . Perhaps the simplest is the Bakry-Émery Ricci tensor, which we denote as $\text{Ric}_X = \text{Ric} + \frac{1}{2}L_X g$. When $X = \nabla f$ for some function f , we write $\text{Ric}_f = \text{Ric} + \text{Hess}f$.

The study of Ric_X pre-dates Bakry-Émery [BÉ85] and goes back at least to Lichnerowicz, who called it the C tensor [Lic70, Lic71]. Ric_X appears in Perelman’s [Pera] and Hamilton’s [Ham95] work on the Ricci flow and is related to Lott-Villani’s [LV09] and Sturm’s [Stu06a, Stu06b] theory of Ricci curvature for metric measure spaces. Partly due to this motivation, a number of mathematicians have recently investigated the connections between Ric_X and the concepts of classical Riemannian geometry. There are far too many recent papers in this direction to reference all of them in this note, so we will only cite the works [Lot03, Mor05, Mor09b, MW12, WW09] as well as chapter 18 of [Mor09a] and the references therein.

The first pinching condition we will consider is a positive lower bound on the Bakry-Émery Ricci curvature and an upper bound on the classical Ricci curvature. After rescaling we will write the condition as

$$(1.1) \quad \text{Ric}_X \geq \varepsilon(n-1)g \quad \text{Ric} \leq (n-1)g \quad \varepsilon > 0.$$

First note that if M is compact, then $\varepsilon \leq 1$ and $\varepsilon = 1$ if and only if M is an Einstein space and X is a Killing field. This follows easily from the divergence theorem. In the compact case we can thus think of the pinching constant ε as measuring how far the space is from being an Einstein manifold with a Killing field.

On the other hand, in the non-compact case it is possible for $\varepsilon > 1$, as the following example shows.

Example 1.1. The Gaussian space is the flat metric on \mathbb{R}^n with $f(x) = \frac{1}{2}|x|^2$. It has $\text{Ric}_f = g$ and $\text{Ric} = 0$. In fact, any simply connected space with bounded

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non-positive sectional curvature has an f such that $\text{Ric}_f \geq g$. See Example 2.2 of [WW09].

On the other hand, all of the manifolds in Example 1.1 are diffeomorphic to \mathbb{R}^n . Our first observation is that this is true in general.

Proposition 1. *Suppose (M, g) is a complete Riemannian manifold and suppose that $\text{Ric}_X > (n - 1)g$ and $\text{Ric} \leq (n - 1)g$; then (M, g) is diffeomorphic to \mathbb{R}^n .*

Note that Proposition 1 is optimal as the product of an Einstein metric equipped with a Killing field and a Gaussian will have $\text{Ric}_X = (n - 1)g$ and $\text{Ric} \leq (n - 1)g$. We also note that there are other interesting spaces satisfying (1.1) as all shrinking Ricci solitons ($\text{Ric}_X = \lambda g$) with bounded Ricci curvature are included in this class. Examples of shrinking Ricci solitons can be found in [Cao96, Koi90, FIK03, WZ04].

We are interested in the topology of spaces satisfying (1.1). In this direction, the author showed in [Wyl08] that if $\text{Ric}_X \geq \varepsilon(n - 1)g$ $\varepsilon > 0$, then M has finite fundamental group. In [FMZ08] it was shown that if $X = \nabla f$ and (1.1) holds, then M is topologically finite, i.e., it is homeomorphic to the interior of a manifold with boundary. Our first main result is to extend this second result to the case where X is not a gradient field.

Theorem 1.2. *Suppose (M, g) is a complete Riemannian manifold admitting a vector field X satisfying (1.1). Then M is homeomorphic to the interior of a manifold with boundary.*

The new ingredient in our proof is to replace Morse theory applied to the potential function f in [FMZ08] with the theory of critical points of the distance function. The proof of Proposition 1 similarly comes from considering critical points of the distance function.

We do not know whether the upper bound on Ricci curvature can be removed from Theorem 1.2. In [FMZ08] it is also shown that a shrinking gradient Ricci soliton is topologically finite if the scalar curvature is bounded.

Myers' diameter estimate states that a complete space with a positive lower bound on the Ricci tensor has diameter less than or equal to the diameter of the sphere of corresponding constant curvature. Non-compact examples like Example 1.1 show Myers' theorem is not true for the spaces satisfying (1.1). On the other hand, we prove the following gap theorem for the diameter of compact spaces satisfying (1.1).

Theorem 1.3. *Suppose that (M, g) is a compact manifold admitting a vector field X satisfying (1.1). Then for any $p \in M$, $M \subset B\left(p, \frac{(n-1)\pi + |X(p)|}{(n-1)\varepsilon}\right)$. In particular, if X has a zero, then $\text{diam}(M) \leq \frac{2\pi}{\varepsilon}$.*

We also obtain a gap theorem for the injectivity radius.

Theorem 1.4. *Suppose that (M, g) is a complete manifold supporting a vector field X which satisfies (1.1). If there is a point $p \in M$ such that $\text{inj}_p(M) \geq \frac{(n-1)\pi + |X(p)|}{(n-1)\varepsilon}$, then M is diffeomorphic to \mathbb{R}^n . In particular, if X has a zero and $\text{inj}(M) \geq \frac{\pi}{\varepsilon}$, then M is diffeomorphic to \mathbb{R}^n .*

We consider these gap theorem results because Example 1.1 shows that there are spaces with infinite diameter and injectivity radius such that for every ε there is a vector field X with a zero satisfying (1.1). Theorems 1.3 and 1.4 show that,

for ε fixed, there cannot be compact spaces satisfying (1.1) with arbitrarily large diameter or injectivity radius.

X will always have a zero when it is the gradient of some function f ; see Corollary 3. On the other hand, an odd dimensional sphere admits a non-trivial constant length Killing field, giving an example satisfying (1.1) where X does not have a zero.

When applied to a gradient shrinking Ricci soliton, these results give bounds on the diameter and injectivity radius in terms of an upper bound on the Ricci curvature. Since this appears to be a new result, we state it as a corollary.

Corollary 1. *If (M, g, f) is a complete gradient shrinking Ricci soliton, $\text{Ric}_f = \lambda g$, with $\text{Ric} \leq \rho$, then either $\text{inj}(M) < \frac{\pi\rho}{\lambda}$ or M is diffeomorphic to \mathbb{R}^n . If M is, in addition, compact, then $\text{diam}(M) \leq \frac{2\rho\pi}{\lambda}$.*

Recently Munteanu and Wang have also established an upper bound on the diameter in terms of the injectivity radius of a gradient Ricci soliton [MW]. Proposition 2.5 of [Web11] also gives an upper bound on the diameter of a compact Ricci soliton in terms of an upper bound on the scalar curvature along with a bound on the constant $C_1 = |\nabla f|^2 + 2\lambda f + \text{scal}$ where f is assumed to be normalized so that $\int_M f = 0$. There is also a universal lower bound on the diameter of a non-Einstein shrinking gradient Ricci soliton [FS13].

In order to obtain finer topological information about curvature pinching, we consider a corresponding version of sectional curvature pinching. While weighted Ricci curvature has been developed extensively, concepts of weighted sectional curvature have been far less studied. In [Wyl] the author introduced two notions of weighted sectional curvature and they have been subsequently studied in [KW]. The notion of weighted sectional curvature we will use in this paper is the following.

Definition 1.5. Suppose (M, g) is a Riemannian manifold and X is a smooth vector field on M . We say that (M, g, X) has a weighted sectional curvature lower bound ε and write $\text{sec}_X \geq \varepsilon$ if for every two-plane σ in $T_p M$ we have

$$\text{sec}(\sigma) \geq \varepsilon - \frac{1}{2}L_X(U, U) \quad \forall U \in \sigma \quad |U| = 1.$$

Note that if $\text{sec}_X \geq \varepsilon$, then $\text{Ric}_{(n-1)X} \geq (n-1)\varepsilon$. Moreover, from the discussion above we see that if a compact manifold satisfies $\text{sec}_X \geq \varepsilon$ and $\text{sec} \leq 1$, then $\varepsilon \leq 1$; $\varepsilon = 1$ if and only if (M, g) has constant curvature and X is a Killing field; and that a complete space with $\text{sec}_X > 1$ and $\text{sec} \leq 1$ is diffeomorphic to \mathbb{R}^n . For examples of metrics with $\text{sec}_X > 0$ but with some negative curvatures, see [KW].

The most famous pinching theorem in Riemannian geometry is the quarter-pinched sphere theorem which states that a simply connected manifold with $1/4 < \text{sec} \leq 1$ is diffeomorphic to the sphere. We are interested in whether there is a similar pinching phenomenon for simply connected manifolds satisfying the condition $\text{sec}_X \geq \varepsilon$ and $\text{sec} \leq 1$. Note that this class of spaces includes the sphere and Euclidean space for all $\varepsilon \leq 1$. We prove a classification up to homotopy equivalence for $\varepsilon > 1/2$.

Theorem 1.6. *Suppose that (M, g) is a simply connected Riemannian manifold and X is a vector field which admits a zero such that $\text{sec}_X \geq \varepsilon > 1/2$ and $\text{sec} \leq 1$.*

- (1) *If M is compact, then M is homeomorphic to a sphere.*
- (2) *If M is complete and non-compact, then M is contractible.*

The quarter-pinched sphere theorem has a long history. The homeomorphism classification goes back to the 1960s and is due to Berger [Ber60] and Klingenberg [Kli61] while the diffeomorphism classification was recently established using Ricci flow techniques by Brendle and Schoen [BS09]. The proof of Theorem 1.6 follows from adapting the classical arguments of Berger and Klingenberg. In fact, the only tool we use is Morse theory for the energy functional, as we do not even have triangle comparison results à la Alexandrov for \sec_X . Because of this, we can only show classification up to homotopy equivalence. In the compact case, the resolution of the Poincaré conjecture then gives the homeomorphism classification. On the other hand, there are contractible manifolds which are not homeomorphic to \mathbb{R}^n , so a homeomorphism classification cannot follow from topological arguments in the non-compact case. We expect, particularly in the non-compact case, that Theorem 1.6 can be improved with further study of the weighted sectional curvature \sec_X .

The paper is organized as follows. In the next section we prove the results about Ricci pinching and review the concept of critical points of the distance function. In section 3 we discuss the examples showing the results from section 2 are optimal. In section 4 we discuss Theorem 1.6.

2. RICCI PINCHING AND CRITICAL POINTS OF THE DISTANCE FUNCTION

The starting point for all of the results of this note is the following estimate involving the second variation of energy formula. It is by no means new and similar results were used, for example, by Hamilton in studying the change in the Riemannian distance along the Ricci flow (See Section 17 of [Ham95]). Also see Proposition 1.94 of [CLN06].

Lemma 2.1. *Suppose (M, g) is a Riemannian manifold and $\gamma : [0, r] \rightarrow M$ is a unit speed minimizing geodesic of length greater than or equal to π . Suppose that $\text{Ric} \leq (n-1)g$ on $B(\gamma(0), \frac{\pi}{2})$ and $B(\gamma(r), \frac{\pi}{2})$. Then*

$$\int_0^r \text{Ric}(\dot{\gamma}, \dot{\gamma}) dt \leq (n-1)\pi.$$

Proof. From the second variation or arclength formula, for a minimal geodesic we have

$$0 \leq \int_0^r (n-1)(\phi')^2 - \phi^2 \text{Ric}(\dot{\gamma}, \dot{\gamma}) dt$$

where ϕ is any function with $\phi(0) = \phi(r) = 0$. This implies that

$$\int_0^r \text{Ric}(\dot{\gamma}, \dot{\gamma}) dt \leq \int_0^r (n-1)(\phi')^2 + (1-\phi^2) \text{Ric}(\dot{\gamma}, \dot{\gamma}) dt.$$

Choose the function

$$\phi(t) = \begin{cases} \sin(t), & 0 \leq t \leq \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} \leq t \leq r - \frac{\pi}{2}, \\ -\sin(t-r), & r - \frac{\pi}{2} \leq t \leq r. \end{cases}$$

Then elementary calculation yields

$$\begin{aligned} & \int_0^r \operatorname{Ric}(\dot{\gamma}, \dot{\gamma}) dt \\ & \leq (n-1) \left(\int_0^{\pi/2} 1 + (\phi')^2 - \phi^2 dt + \int_{r-\frac{\pi}{2}}^r 1 + (\phi')^2 - \phi^2 dt \right) \\ & = (n-1) \left(\int_0^{\pi/2} 1 + \cos^2(t) - \sin^2(t) dt + \int_{r-\frac{\pi}{2}}^r 1 + \cos^2(t-r) - \sin^2(t-r) dt \right) \\ & = (n-1)\pi. \end{aligned}$$

□

Under our Ricci pinching assumption this gives us the following estimate.

Lemma 2.2. *Let (M, g) be a complete Riemannian manifold that admits a vector field X satisfying (1.1). Suppose that $p, q \in M$ and let γ be a unit speed minimal geodesic from p to q . Then*

$$g(X(q), \dot{\gamma}(d(p, q))) \geq -(n-1)\pi - |X(p)| + (n-1)\varepsilon d(p, q).$$

Proof. We have

$$\frac{1}{2} L_X g(\dot{\gamma}, \dot{\gamma}) = g(\nabla_{\dot{\gamma}} X, \dot{\gamma}) = \frac{d}{dt} g(X, \dot{\gamma})$$

where $\frac{d}{dt}$ denotes the derivative along the geodesic. If we integrate the equation $\operatorname{Ric} + \frac{1}{2} L_X g \geq \varepsilon(n-1)g$ along the geodesic we obtain,

$$\int_0^{d(p,q)} \operatorname{Ric}(\dot{\gamma}, \dot{\gamma}) dt + g(X(q), \dot{\gamma}(d(p, q))) - g(X(p), \dot{\gamma}(0)) \geq (n-1)\varepsilon d(p, q).$$

Applying Lemma 2.1 then gives us the formula

$$g(X(q), \dot{\gamma}(d(p, q))) \geq -(n-1)\pi - |X(p)| + (n-1)\varepsilon d(p, q).$$

□

Two corollaries of this formula that we will find useful are the following. The second is already stated in [FMZ08] and also appears implicitly as part of the proof of Lemma 1.2 of [Perb]. We include the proofs for completeness.

Corollary 2. *If (M, g) is a complete non-compact manifold that admits a vector field X satisfying (1.1), then $|X| \rightarrow \infty$ at infinity.*

Proof. Fix p and let $q \rightarrow \infty$; then we obtain that $g(X(q), \dot{\gamma}(d(p, q))) \rightarrow \infty$ so that, in particular, $|X| \rightarrow \infty$. □

Corollary 3. *If (M, g) is a complete non-compact manifold that admits a gradient vector field $X = \nabla f$ satisfying (1.1), then f grows at least quadratically with the distance to any point. In particular, f has at least one critical point as it obtains its minimum.*

Proof. When $X = \nabla f$, for any geodesic $\gamma(t)$ we have

$$\begin{aligned} \frac{d}{dt} (f(\gamma(t))) & = g(X(\gamma(t)), \dot{\gamma}(t)) \\ & \geq -(n-1)\pi - |X(\gamma(0))| + (n-1)\varepsilon t. \end{aligned}$$

Integrating the equation along γ then gives the result. □

Remark 2.3. In the case of a gradient Ricci soliton Cao-Zhou show that f grows exactly quadratically [CZ10].

In order to obtain topological results from these formulae we will use the theory of critical points of the distance function which was pioneered by Grove-Shiohama in [GS77]. There are many surveys of the theory. For completeness we will review the main elements and refer the reader to, for example, Chapter 11 of [Pet06] for a more complete treatment.

Recall that for a smooth function $f : M \rightarrow \mathbb{R}$, the critical points are the points where $df = 0$. If f is proper and there are no critical points in $f^{-1}([a, b])$ one of the foundational lemmas of Morse theory says that $f^{-1}((-\infty, b])$ deformation retracts onto $f^{-1}((-\infty, a])$. Thus, in the case $X = \nabla f$, Lemma 2.2 already shows that f is proper and has no critical points outside of some compact set.

Let $p \in M$ and consider the distance function r to p . r is smooth almost everywhere with unit gradient so it does not have critical points in the traditional sense. The approach to defining critical points for r is to view critical points as the obstruction to producing a deformation retraction between sub-levels. At the points where r is smooth, ∇r is the unit tangent vector to the unique minimal geodesic from p to the point. At the points where r is not smooth, there may be multiple minimal geodesics from p to the point. Intuitively, it still might be possible to deformation retract past these points if all these geodesics point in roughly the same direction. Indeed, it turns out all we need to build the retract is that all of the tangent vectors of the minimal geodesics from p to q lie in some half space of $T_q M$. This gives the following definition of a critical point.

Definition 2.4. Fix $p \in M$. A point q is a critical point of the distance function to p (is critical to p) if, for every vector $V \in T_q M$, there is a minimal geodesic γ with $\gamma(0) = p$, $\gamma(d(p, q)) = q$ such that $g(\dot{\gamma}(d(p, q)), V) \leq 0$.

In all of the results of this paper, we'll show that certain points q are not critical to p by showing that for every minimal geodesic from p to q , $g(\dot{\gamma}(d(p, q)), X(q)) > 0$. In particular, our arguments do not have an analogue in the classical case where $X = 0$.

As alluded to in the discussion above, we have the following topological lemma about critical points of the distance function. See page 337 of [Pet06].

Lemma 2.5. *Suppose that there are no critical points of the distance function to p in the annulus $\{q : a \leq d(p, q) \leq b\}$. Then $B(p, a)$ is homeomorphic to $B(p, b)$ and $B(p, b)$ deformation retracts onto $B(p, a)$. Moreover, if there are no critical points of p in M , then M is diffeomorphic to \mathbb{R}^n .*

Remark 2.6. We cannot get diffeomorphic sub-levels in general because the levels of r will not in general be smooth. This is not an issue when there are no critical values because we can retract down to a small neighborhood where the levels are smooth.

Our first application is Proposition 1.

Proof of Proposition 1. From the assumptions we have $L_X g > 0$. From the divergence theorem this implies the manifold is non-compact. From Corollary 2 we have $|X|^2 \rightarrow \infty$ at infinity which implies that the function $\phi(p) = |X(p)|^2$ has a minimum somewhere on M . At the minimum point we must have

$$0 = D_X \phi = D_X g(X, X) = 2L_X g(X, X).$$

Since $L_X g > 0$ this implies the minimum of ϕ must be zero. Let $\gamma(t)$ be a geodesic with $\gamma(0) = p$ a zero of X . Then we obtain

$$0 < \int_0^t L_X g(\dot{\gamma}, \dot{\gamma}) dt = g(X(\gamma(t)), \dot{\gamma}(t))$$

which shows that there are no critical points to p and thus the manifold is diffeomorphic to Euclidean space. □

Remark 2.7. We note that the last part of the proof of Proposition 1 proves the following: *If M is a complete manifold supporting a vector field with a zero such that $L_X g > 0$, then M is diffeomorphic to Euclidean space.* We note that even in the gradient case this is slightly different from the result one would obtain from classical Morse theory as we do not need to assume that f is proper but have added the assumption that the metric is complete. On the other hand, the assumption that X has a zero is necessary. A simple example is to let $g = dt^2 + e^{2t} g_N$ be the warped product metric on $\mathbb{R} \times N$ and define $f = e^t$, which has $\text{Hess}_g f > 0$.

More generally, we have the following critical point estimate.

Lemma 2.8. *Let (M, g) be a complete Riemannian manifold admitting a vector field X which satisfies (1.1). Let $p \in M$. If $q \in M$ satisfies*

$$d(p, q) > \frac{(n - 1)\pi + |X(p)|}{(n - 1)\varepsilon},$$

then q is not critical to p .

Proof. If $d(p, q) > \frac{(n-1)\pi+|X(p)|}{(n-1)\varepsilon}$, then by Lemma 2.2, $g(X(q), \dot{\gamma}(d(p, q))) > 0$. Since this is true for all minimal geodesics γ from p to q , this shows that q is not critical to p . □

This immediately gives us Theorem 1.2.

Proof of Theorem 1.2. Fix p . Lemma 2.8 tells us that there are no critical points to p outside of a compact set. Thus the manifold deformation retracts onto a finite open ball in M , showing it is homeomorphic to the interior of a manifold with boundary. □

Now we consider the diameter estimate. Unlike the results above, this result does not seem to have been observed before even in the gradient case. The main “new” ingredient is the following fact about critical points of the distance function due to Berger [Ber60].

Lemma 2.9 ([Ber60]). *Let (M, g) be a compact Riemannian manifold and fix $p \in M$. Let q be a point such that $d(p, q) = \max_{x \in M} d(p, x)$. Then q is critical to p .*

Remark 2.10. This result is often stated for p and q which realize $\text{diam}(M, g) = d(p, q)$, but the same proof works in this case. See [dC92] p. 283.

The proofs of Theorems 1.3 and 1.4 are now immediate from Berger’s Lemma and Lemma 2.8.

Proof of Theorem 1.3. By Lemma 2.9, if q is a furthest point from p , then q must be critical to p . Then by Lemma 2.8, $d(p, q) \leq \frac{(n-1)\pi+|X(p)|}{(n-1)\varepsilon}$. □

Proof of Theorem 1.4. From the hypothesis on injectivity radius there are no critical points of the distance function inside $B\left(p, \frac{(n-1)\pi+|X(p)|}{(n-1)\varepsilon}\right)$. Lemma 2.8 tells us there are also no critical points outside of the ball so the manifold is diffeomorphic to \mathbb{R}^n . \square

3. EXAMPLE

In this section we construct examples showing the gap theorems in the previous section are optimal. We have the following proposition.

Proposition 2. *Let $n \geq 3$. For any $\varepsilon < \frac{n-1}{n-2}$ there is a family of Riemannian manifolds with density on the n -sphere (g_δ, f_δ) defined for δ sufficiently small with $\text{Ric}_{g_\delta, f_\delta} \geq (n-1)\varepsilon$, $\text{Ric}_{g_\delta} \leq (n-1)$ and such that there is a point p with $\sup_{q \in S^n} d_{g_\delta}(p, q) = \text{inj}_p(g_\delta) = L_\delta$ where $L_\delta \rightarrow \frac{\pi}{\varepsilon}$ as $\delta \rightarrow 0$.*

Proof. The metrics are rotationally symmetric metrics of the form

$$dr^2 + \phi^2(r)g_{S^{n-1}}.$$

Let ∂_r denote the tangent vector in the r direction and let X be a unit vector perpendicular to ∂_r . We will define $f = f(r)$ our potential function to be a function of r . Then the Bakry-Émery tensor has the formula

$$\begin{aligned} \text{Ric}_f(\partial_r, \partial_r) &= -(n-1)\frac{\ddot{\phi}}{\phi} + \ddot{f} \\ \text{Ric}_f(\partial_r, X) &= 0 \\ \text{Ric}_f(X, X) &= -\frac{\ddot{\phi}}{\phi} + (n-2)\left(\frac{1 - (\dot{\phi})^2}{\phi^2}\right) + \frac{f\dot{\phi}}{\phi}. \end{aligned}$$

Consider a C^2 half-capped cylinder given by ϕ defined on $[0, L]$ as

$$\phi(r) = \begin{cases} \sin(r), & 0 \leq r < \frac{\pi}{2} - \delta, \\ \phi_0(r), & \frac{\pi}{2} - \delta \leq r < \frac{\pi}{2}, \\ A, & \frac{\pi}{2} \leq r \leq L, \end{cases}$$

where $\phi_0(r)$ will be described later, and L , δ and A are positive numbers to be chosen. For our potential function we define

$$f(r) = \begin{cases} -\frac{n-1}{2}(1-\varepsilon)r^2 & 0 \leq r < \frac{\pi}{2} - 2\delta \\ f_0(r) & \frac{\pi}{2} - 2\delta \leq r < \frac{\pi}{2} - \delta \\ \frac{(n-1)\varepsilon}{2}(r - \frac{\pi}{2} + \delta)^2 - (n-1)(1-\varepsilon)(\frac{\pi}{2} - 2\delta)(r - \frac{\pi}{2} + \delta) + B & \frac{\pi}{2} - \delta \leq r \leq L \end{cases}$$

where the function f_0 will be described later and B is another constant to be chosen.

For $r > \frac{\pi}{2} - \delta$, f is a parabola with a critical point at the point $\frac{\pi}{2\varepsilon} + \delta + 2\frac{\delta}{\varepsilon}$ and the metric is a flat cylinder. Thus we can double f and ϕ to obtain a smooth metric on the sphere of diameter and injectivity radius, $L_\delta = \frac{\pi}{\varepsilon} + 2\delta + 4\frac{\delta}{\varepsilon}$.

For $r \in [0, \frac{\pi}{2} - 2\delta]$ we have

$$\begin{aligned} \text{Ric}_f(\partial_r, \partial_r) &= (n-1)\varepsilon, \\ \text{Ric}_f(X, X) &= (n-1)(1 - (1-\varepsilon)r \cot(r)), \\ &\geq (n-1)\varepsilon \end{aligned}$$

since $r \cot(r)$ decreases from 1 to zero on $[0, \frac{\pi}{2}]$. For $r \geq \frac{\pi}{2}$ we have

$$\begin{aligned} \text{Ric}_f(\partial_r, \partial_r) &= (n-1)\varepsilon, \\ \text{Ric}_f(X, X) &= (n-2)(1/A). \end{aligned}$$

Thus, as long as $\varepsilon < \frac{n-2}{(n-1)A}$, we have $\text{Ric}_f \geq (n-1)\varepsilon g$ outside of region $\frac{\pi}{2} - 2\delta \leq r < \frac{\pi}{2}$. To control the curvature in that region we must choose ϕ_0 and f_0 appropriately. In order for f to be smooth at $\pi/2 - 2\delta$ we choose f_0 to satisfy the initial conditions

$$\ddot{f}_0\left(\frac{\pi}{2} - 2\delta\right) = -(n-1)(1-\varepsilon) \quad \& \quad \dot{f}_0\left(\frac{\pi}{2} - 2\delta\right) = -(n-1)(1-\varepsilon)\left(\frac{\pi}{2} - 2\delta\right).$$

Define f_0 by prescribing $\dot{f}_0(r)$ to be a monotone increasing function that satisfies $\dot{f}_0(\pi/2 - \delta) = (n-1)\varepsilon$ and $\int_{\pi/2-2\delta}^{\pi/2-\delta} \ddot{f}_0(t)dt = 0$. The second condition assures that f will also be smooth at $\pi/2 - \delta$ after choosing C to be an appropriate constant.

Then, for any $r \in [\pi/2 - 2\delta, \pi/2 - \delta]$, we have

$$\begin{aligned} \dot{f}_0(r) &= \int_{\pi/2-2\delta}^r \ddot{f}_0(t)dt + \dot{f}_0(\pi/2 - 2\delta) \\ &\geq -(n-1)(1-\varepsilon)\left(\frac{\pi}{2} - \delta\right). \end{aligned}$$

For the curvature we then have

$$\begin{aligned} \text{Ric}_f(\partial_r, \partial_r) &\geq (n-1)\varepsilon, \\ \text{Ric}_f(X, X) &= (n-1) + \dot{f}(r) \cot(r) \\ &\geq (n-1)(1 - (1-\varepsilon)(\pi/2 - \delta) \cot(r)). \end{aligned}$$

Since $\cot(r) \rightarrow 0$ at $\pi/2$ we can thus see that $\text{Ric}_f(X, X) \geq (n-1)\varepsilon$ if δ is chosen small enough.

Define ϕ_0 similarly by assuming it satisfies the initial conditions

$$\phi_0\left(\frac{\pi}{2} - \delta\right) = \sin\left(\frac{\pi}{2} - \delta\right) \quad \& \quad \dot{\phi}_0\left(\frac{\pi}{2} - \delta\right) = \cos\left(\frac{\pi}{2} - \delta\right) \quad \& \quad \ddot{\phi}_0\left(\frac{\pi}{2} - \delta\right) = -\sin\left(\frac{\pi}{2} - \delta\right)$$

and letting $\ddot{\phi}$ increase from $-\sin(\frac{\pi}{2} - \delta)$ to 0 such that it satisfies $-\cos(\pi/2 - \delta) = \int_{\pi/2-\delta}^{\pi/2} \ddot{\phi}(t)dt$. Then, choosing A to be an appropriate constant, we will have a C^2 function ϕ . When δ is small, A will be close to 1 and we can see that ϕ_0 will be close to A , $\dot{\phi}_0$ is close to zero and $\ddot{\phi} \geq 0$, which implies that $\text{Ric}_f \geq (n-1)\varepsilon$ in this region as well. □

4. PINCHED SECTIONAL CURVATURE

Now we fix some more notation for the weighted sectional curvature. Given an orthonormal pair of vectors U, V we define

$$\text{sec}_X(U, V) = \text{sec}(U, V) + \frac{1}{2}L_X g(U, U)$$

where $\text{sec}(U, V)$ is the sectional curvature of the plan spanned by U and V . Note that the weighted curvature $\text{sec}_X(U, V)$ is not symmetric in U and V . See Section 2 of [Wy1] for further discussion.

As is traditional, the topological tool we will use here is not critical points of the distance function, but the Morse theory of the energy functional applied to the path space. For submanifolds A and B in M define the path space as

$$\Omega_{A,B}(M) = \{\gamma : [0, 1] \rightarrow M, \gamma(0) = A, \gamma(1) = B\}.$$

We will only look at the space of paths between points p and q , $\Omega_{p,q}$. We consider the energy $E : \Omega_{p,q}(M) \rightarrow \mathbb{R}$ and variation fields which vanish at both end points of the curve. The critical points of E are then the geodesics connecting p and q and we say the index of such a geodesic is $\geq k$ if there is a k -dimensional space of

variation fields along the geodesic which have negative second variation. In order to estimate the index of a geodesic, we have the following lemma for sectional curvature modeled on Lemma 2.1.

Lemma 4.1. *Suppose (M, g) is a Riemannian manifold with $\sec \leq 1$ and $\gamma : [0, r] \rightarrow M$ is a unit speed minimizing geodesic of length greater than or equal to π . Suppose that for any perpendicular parallel field along γ , E , we have*

$$\int_0^r \sec(\dot{\gamma}, E) dt > \pi.$$

Then the index of γ is greater than or equal to $(n - 1)$.

Proof. Let V be the proper variation field ϕE where ϕ is the function constructed in Lemma 2.1. The second variation of arc length formula shows that

$$\frac{d^2}{ds^2} E(0) = \int_0^r (\phi')^2 - \phi^2 \sec(\dot{\gamma}, E) dt.$$

Assume that $\frac{d^2}{ds^2} E(0) \geq 0$; then imitating the proof of Lemma 2.1 implies that $\int_\gamma \sec(\dot{\gamma}, E) dt \leq \pi$. Therefore if $\int_0^r \sec(\dot{\gamma}, E) dt > \pi$, then the second variation must be negative, since there are $(n - 1)$ linearly independent fields, and we obtain the result. □

Applying the lemma to the elements of $\Omega_{p,p}$ gives us the following.

Lemma 4.2. *Suppose that M is a complete Riemannian manifold supporting a vector field X with a zero at a point p such that*

$$\sec_X \geq \varepsilon \quad \text{and} \quad \sec \leq 1 \quad 0 < \varepsilon < 1.$$

If γ is a geodesic loop based at p of length greater than $\frac{\pi}{\varepsilon}$, then $\text{index}(\gamma) \geq (n - 1)$.

Proof. Let E be a unit parallel field around γ . Then

$$\begin{aligned} \varepsilon \text{length}(\gamma) &\leq \int_0^r \sec_X(\dot{\gamma}, E) dt = \int_0^r \sec(\dot{\gamma}, E) + \frac{d}{dt} g(X, \dot{\gamma}) dt \\ &= \int_0^r \sec(\dot{\gamma}, E) dt. \end{aligned}$$

Therefore, for such a geodesic, $\int_0^r \sec(\dot{\gamma}, E) dt > \pi$ for all unit perpendicular parallel fields. By the previous lemma, $\text{index}(\gamma) \geq (n - 1)$. □

Remark 4.3. In the example in the previous section, if we let γ be the geodesic in the ∂_r directions we have $\sec_f(\partial_r, E) \geq \varepsilon$, $\sec \leq 1$, and we can make γ minimizing for a length arbitrarily close to $\frac{\pi}{\varepsilon}$ by taking $\delta \rightarrow 0$. This shows the proof of Lemma 4.2 is optimal. On the other hand, the examples of section 2 do not have $\sec_X > 0$ because if U, V are both perpendicular to ∂_r on the region where $r > \pi/2$, $\sec_X(U, V) = 0$. For examples of compact metrics with $\sec_X > 0$ and $\sec \leq 1$ which do not have $\sec > 0$ we refer the reader to [KW].

This gives us the following generalization of a sphere theorem of Berger [Ber58].

Theorem 4.4. *Suppose that M is a complete Riemannian manifold supporting a vector field X with a zero at a point p such that*

$$\sec_X > \varepsilon \quad \text{and} \quad \sec \leq 1 \quad 0 < \varepsilon < 1.$$

If $\text{inj}_p(M) > \frac{\pi}{2\varepsilon}$, then M is either homotopic to a sphere or it is contractible.

Proof. If $\text{inj}_p(M) > \frac{\pi}{2\varepsilon}$, then every geodesic in $\Omega_{p,p}$ has length greater than $\frac{\pi}{\varepsilon}$. From the previous lemma, this shows that every geodesic must have index at least $(n - 1)$. This implies (see Theorem 32 of [Pet06]) that M is $(n - 1)$ -connected.

An $(n - 1)$ -connected compact n -manifold is homotopic to the sphere. To see this, first note that by the Hurewicz Theorem we have $\pi_n(M) = H_n(M)$. In the compact case, by Poincaré duality, we have $H_n(M) = \mathbb{Z}$ so there is a degree 1 map from M to the sphere, and this map induces isomorphisms on $H_i(M)$ for all i . By a theorem of Whitehead (see page 418 of [Hat02]) a map inducing isomorphisms on all homology groups is a homotopy equivalence.

By a similar topological argument, an $(n - 1)$ -connected non-compact n -manifold is contractible. This follows because now we have $H_n(M) = 0$, and a map from a point into M then induces isomorphisms on all homology groups and is thus a homotopy equivalence. □

Theorem 4.4 shows that all we need is an injectivity radius estimate in order to prove Theorem 1.6. In the classical case, the injectivity radius estimate under $1/4$ -pinching is due to Klingenberg [Kli61]. Our injectivity radius estimate is similar to Klingenberg's; we refer the reader to the texts [dC92, Kli82, Pet06] for the details of the proof. The first step is the long homotopy lemma, which only depends on an upper bound on curvature, so extends immediately to our setting.

Lemma 4.5 (Klingenberg's Long Homotopy Lemma). *Suppose that we have $\text{sec} \leq 1$ and suppose that $p, q \in M$ such that p and q are joined by two distinct geodesics γ_0 and γ_1 which are homotopic by a homotopy α_t . Then there exists a curve in the homotopy α_{t_0} such that*

$$\text{length}(\alpha_{t_0}) \geq 2\pi - \min\{\text{length}(\gamma_i)\}.$$

The proof of the injectivity radius estimate now follows the same general argument as in the classical case. The only difference is that we use Lemma 4.1 to control the index. We refer the reader to p. 279 of [dC92] for the version of the classical proof on which ours is modeled.

Theorem 4.6. *Suppose that (M, g, X) is complete, simply connected, and has*

$$\text{sec}_X \geq \varepsilon > \frac{1}{2} \quad \text{and} \quad \text{sec} \leq 1.$$

If p is a zero of X , then $\text{inj}_p(M) \geq \pi$.

Proof. If $\text{inj}_p(M) < \pi$, then there is a geodesic loop γ based at p of length $l < 2\pi$. Choose δ such that

- (1) $\gamma(l - \delta)$ is not conjugate to p ,
- (2) \exp_p is a diffeomorphism on $B_p(2\delta)$,
- (3) $3\varepsilon\delta + N(\delta) < \pi(2\varepsilon - 1)$ where $N(\delta) = \max\{|X(q)| : q \in B_p(2\delta)\}$,
- (4) $3\delta < 2\pi - l$, and
- (5) $5\delta < 2\pi$.

From Sard's theorem there exists a regular value $q \in B_{\gamma(l-\delta)}(\delta)$ of \exp_p . Since $\gamma(l - \delta)$ is not conjugate to p it is possible to choose a geodesic γ_1 starting at p to q with $3\delta < \text{length}(\gamma_1) < l$.

Let γ_0 be the minimizing geodesic from p to q ; then $\text{length}(\gamma_0) \leq 2\delta$. In particular, $\gamma_0 \neq \gamma_1$. Since M is simply connected there is a homotopy γ_t from γ_0 to γ_1 .

From the Long Homotopy Lemma we have that any such homotopy must contain a curve of length at least $2\pi - \min\{\text{length}(\gamma_i)\}$. Since we have $l(\gamma_0) < 2\delta$ and $l(\gamma_1) < l < 2\pi - 3\delta$, the condition $5\delta \leq 2\pi$ implies that we must have a curve of length at least $2\pi - 2\delta$ in the homotopy.

Now consider Morse theory applied to the space $\Omega_{p,q}$ with E as a Morse function. Since q is a regular value of \exp_p , we have that all of the critical points (geodesics) are non-degenerate. From Morse theory, if there was no geodesic with index less than two and length at least $2\pi - 3\delta$, then we could push the homotopy from γ_0 to γ_1 into the region where $E < 2\pi - 2\delta$ and contradict the Long Homotopy Lemma. Therefore, we must have a geodesic from p to q with index zero or one which has length r greater than $2\pi - 3\delta$. However, for a parallel field, E along this geodesic σ we have

$$\varepsilon r \leq \int_0^r \sec_X(\dot{\sigma}, E) dt = \int_0^r \sec(\dot{\sigma}, E) dt + g(X, \dot{\sigma})(q) \leq \int_0^r \sec(\dot{\sigma}, E) dt + N(\delta).$$

So we have

$$\varepsilon(2\pi - 3\delta) - N(\delta) \leq \int_0^r \sec(\dot{\sigma}, E) dt.$$

Condition (3) then implies that $\int_0^r \sec(\dot{\sigma}, E) dt > \pi$ for every unit parallel field along σ , showing that the index of σ is $n - 1$, a contradiction to the assumption $\text{inj}_p(M) < \pi$. \square

Combining Theorems 4.4 and 4.6 now gives Theorem 1.6. A natural question is whether the $1/2$ pinching constant is optimal in Theorem 1.6. We note that appearance of the $1/2$ as opposed to the $1/4$ in the classical case comes from the difference in the index estimate (Lemma 4.1). We know from the examples in section 3 that Lemma 4.1 is optimal, so this at least shows that the method of this paper is not sufficient to improve the pinching constant.

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DEPARTMENT OF MATHEMATICS, 215 CARNEGIE BUILDING, SYRACUSE UNIVERSITY, SYRACUSE,
NEW YORK 13244

E-mail address: wylie@syr.edu

URL: <https://wylie.expressions.syr.edu>