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Conformal Diffeomorphisms of Gradient Ricci Solitons and Generalized Quasi-Einstein Manifolds

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Abstract In this paper we extend some well-known rigidity results for conformal changes of Einstein metrics to the class of generalized quasi-Einstein (GQE) metrics, which includes gradient Ricci solitons. In order to do so, we introduce the notions of conformal diffeomorphisms and vector fields that preserve a GQE structure. We show that a complete GQE metric admits a structure-preserving, non-homothetic complete conformal vector field if and only if it is a round sphere. We also classify the structure-preserving conformal diffeomorphisms. In the compact case, if a GQE metric admits a structure-preserving, non-homothetic conformal diffeomorphism, then the metric is conformal to the sphere, and isometric to the sphere in the case of a gradient Ricci soliton. In the complete case, the only structure-preserving non-homothetic conformal diffeomorphisms from a shrinking or steady gradient Ricci soliton to another soliton are the conformal transformations of spheres and inverse stereographic projection.

Keywords Conformal diffeomorphism · Conformal Killing field · Generalized quasi Einstein space · Gradient Ricci soliton · Warped product

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1 Introduction

It is well known that the Einstein condition on a Riemannian manifold is not conformally invariant. In the 1920s Brinkmann [2] classified when two Einstein metrics are conformal to each other and Yano–Nagano [32] later proved that if a complete Einstein metric admits a complete conformal field then it is a round sphere. For further results in this direction, see [23, 25], and pp. 309–311 of [18]. For the pseudo-Riemannian case and many more references, see [22].

In this paper we show that these results have natural extensions to the class of *generalized quasi-Einstein* (GQE) metrics, that is, Riemannian metrics g on a manifold M of dimension $n \geq 3$ satisfying

$$\mathrm{Ric} + \mathrm{Hess} f + \alpha df \otimes df = \lambda g \tag{1.1}$$

for some smooth functions f, α, λ on M, where Ric and Hess are the Ricci curvature and Hessian with respect to g. GQE manifolds[1] were recently introduced by Catino [7], who proved a local classification of GQE metrics with divergence-free Weyl tensor. GQE metrics generalize:

- Einstein metrics: $\mathrm{Ric} = \lambda g$ where $\lambda \in \mathbb{R}$,
- gradient Ricci solitons: $\mathrm{Ric} + \mathrm{Hess} f = \lambda g$, where $\lambda \in \mathbb{R}$,
- gradient Ricci *almost* solitons: $\mathrm{Ric} + \mathrm{Hess} f = \lambda g$, where $\lambda \in C^\infty(M)$, introduced by Pigola–Rigoli–Rimoldi–Setti [29], and
- m-quasi-Einstein metrics: $\alpha = -\frac{1}{m}$ for a positive integer m and $\lambda \in \mathbb{R}$, introduced by Case–Shu–Wei [6]; these include the static metrics when $m = 1$.

We will consider diffeomorphisms between GQE manifolds that preserve the structure in the following sense.

Definition 1.1 A diffeomorphism ϕ from a GQE manifold $(M_1, g_1, f_1, \alpha_1, \lambda_1)$ to a GQE manifold $(M_2, g_2, f_2, \alpha_2, \lambda_2)$ is said to *preserve the GQE structure* if $\phi^*\alpha_2 = \alpha_1$ and $\phi^* df_2 = df_1$. A vector field V on a GQE manifold *preserves the GQE structure* if $D_V \alpha = 0$ and $D_V f$ is constant, or equivalently, if the local flows of V preserve the GQE structure.

A *conformal diffeomorphism* ϕ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is a diffeomorphism such that

$$\phi^* g_2 = w^{-2} g_1$$

for some function $w > 0$ on M_1. A *conformal vector field* on a Riemannian manifold (M, g) is a vector field whose local flows are conformal diffeomorphisms; equivalently, V satisfies

$$L_V g = 2\sigma g$$

for some function σ on M, where L is the Lie derivative.

[1] We note that this class of metrics differs from the Kähler generalized quasi-Einstein metrics of Guan [13] and the generalized quasi-Einstein metrics of Chaki [8].

A trivial example of a conformal diffeomorphism that preserves the GQE structure is any homothetic rescaling (ϕ = identity, $g_2 = c^2 g_1$). We will say a conformal diffeomorphism is *non-homothetic* if w is not constant. Similarly, a conformal vector field is non-homothetic if σ is not constant.

We show that conformal diffeomorphisms and vector fields that preserve a GQE structure only exist in very rigid situations. Our most general result is the following classification theorem for non-homothetic conformal transformations that preserve a generalized quasi-Einstein structure. This result holds in both the local and global settings.

Theorem 1.2 *Let ϕ be a non-homothetic, structure-preserving conformal diffeomorphism between GQE manifolds $(M_1, g_1, f_1, \alpha_1, \lambda_1)$ and $(M_2, g_2, f_2, \alpha_2, \lambda_2)$ of dimension $n \geq 3$. Then, about points where $\alpha_i \neq \frac{1}{n-2}$, g_1 and g_2 are both of the form*

$$g_i = ds^2 + v_i(s)^2 g_N, \tag{1.2}$$

where (N, g_N) is an $(n-1)$-manifold independent of s and $f_i = f_i(s)$, or

$$g_i = e^{\frac{2f_i}{n-2}}\left(ds^2 + v_i(s)^2 g_N\right), \tag{1.3}$$

where (N, g_N) is an $(n-1)$-manifold independent of s and f_i is a function on N. If either g_1 or g_2 is complete and $\alpha_i \neq \frac{1}{n-2}$, then the metrics are globally either of the form (1.2) or (1.3). Moreover, in case (1.2), if $n \geq 4$ or α_1 is constant, then g_N is Einstein; in case (1.3), g_N is conformal to a GQE manifold with potential f_i. Finally, only (1.2) is possible if $n = 3$.

Remark 1.3 If $\alpha_1 \equiv \frac{1}{n-2}$, then g_1 is conformal to an Einstein metric (see Proposition 3.1); these spaces fall into the Einstein case studied by Brinkmann [2].

Remark 1.4 When f is constant, cases (1.2) and (1.3) are the same. The metric g_2 need not be complete if g_1 is, even in the Einstein case; stereographic projection provides a counterexample.

Remark 1.5 We give examples in Sects. 4.1 and 4.2 showing both cases in Theorem 1.2 may occur. We also show that the two cases do not occur on the same connected manifold unless f is constant.

In the compact case, we further obtain the following.

Theorem 1.6 *Let ϕ be a non-homothetic, structure-preserving conformal diffeomorphism between compact GQE manifolds $(M_1, g_1, f_1, \alpha_1, \lambda_1)$ and $(M_2, g_2, f_2, \alpha_2, \lambda_2)$. Then (M_i, g_i) are conformally diffeomorphic to the standard round metric on S^n. Moreover, if $\alpha_1 \neq \frac{1}{n-2}$, then (M_i, g_i) are rotationally symmetric.*

The case of conformal fields exhibits greater rigidity than the case of discrete conformal changes. For instance, we prove:

Theorem 1.7 *Suppose* $(M, g, f, \alpha, \lambda)$ *is a complete GQE manifold, with* $\alpha \neq \frac{1}{n-2}$, *that admits a structure-preserving non-homothetic conformal field*: $L_V g = 2\sigma g$. *If* σ *has a critical point* (*e.g., if* M *is compact*), *then* f *is constant and* (M, g) *is isometric to a simply connected space form.*

Moreover, the round sphere is the only possibility if the conformal field is assumed to be complete, generalizing Yano–Nagano's result.

Theorem 1.8 *If a complete GQE manifold* $(M, g, f, \alpha, \lambda)$ *with* $\alpha \neq \frac{1}{n-2}$ *admits a non-homothetic complete conformal field* V *preserving the GQE structure then* f *is constant and* (M, g) *is isometric to a round sphere.*

In fact, we obtain a full local classification without the completeness assumption on V or g. There are several examples; we delay further discussion to Sect. 5.

We also obtain more rigidity in the case of a gradient Ricci soliton (i.e., $\alpha = 0$ and λ is constant). A gradient Ricci soliton (M, g, f) is called *shrinking*, *steady*, or *expanding* depending on whether $\lambda > 0$, $\lambda = 0$, $\lambda < 0$ respectively. Combining our results with some other well-known results for gradient Ricci solitons gives us the following theorem.

Theorem 1.9 *Let* ϕ *be a non-homothetic, structure-preserving conformal diffeomorphism between GQE manifolds* $(M_1, g_1, f_1, 0, \lambda_1)$ *and* $(M_2, g_2, f_2, 0, \lambda_2)$, *and assume* (M_1, g_1, f_1) *is a complete gradient Ricci soliton. Then* g_1 *and* g_2 *are both metrics of the form* (1.2), *and*:

- *If* M_1 *is compact, then* g_1 *and* g_2 *are both round metrics on the sphere.*
- *If* (M_1, g_1, f_1) *is either shrinking or steady, then it is a round metric on the sphere, the flat metric on* $\mathbb{R}^n$, *the Bryant soliton, or a product* $\mathbb{R} \times N$ *where* N *is Einstein with Einstein constant* λ.
- *If, in addition,* (M_2, g_2, f_2) *is a soliton, then either* (M_i, g_i) *are round metrics on the sphere or* ϕ *is an inverse stereographic projection with* (M_1, g_1) *flat Euclidean space and* (M_2, g_2) *a round spherical metric with a point removed.*
- *If* (M, g, f) *is a complete gradient Ricci soliton admitting a non-homothetic conformal field that preserves the structure, then* (M, g) *is Einstein and* f *is constant.*

Remark 1.10 In the last case, note that complete Einstein metrics admitting non-homothetic conformal fields were classified by Kanai [16]; we recall this classification in Remark 6.4.

Remark 1.11 We also obtain that g_1 and g_2 are both of the form (1.2) when g_1 is m-quasi-Einstein. m-quasi-Einstein metrics of the form (1.2) are found in [1] (cf. [14]). Examples of complete expanding gradient Ricci solitons of the form (1.2) are found in [11].

Remark 1.12 Interesting results for some conformal changes of Kähler Ricci solitons that do not preserve the GQE structure are obtained in [24].

Remark 1.13 The results of this paper can be viewed as an initial investigation into the interesting more general problem of understanding when two generalized quasi-Einstein metrics (or gradient Ricci solitons) can be related by a conformal change. Further investigations into the case where the GQE structure is not preserved could potentially yield new constructions of examples or a more general classification.

The paper is organized as follows. In Sect. 2 we discuss warped product metrics with a one-dimensional base. The first observation, known to Brinkmann, is a characterization of these spaces as those admitting a non-constant solution to a certain PDE. The second observation is a duo of completeness lemmas for metrics conformal to a warped product, in which the conformal factor is either a function on only the base or only the fiber. Section 3 is the technical heart of the paper. We recast the GQE condition on g in terms of an equivalent condition on a conformally rescaled metric h, then establish a warped product structure on h. Next, we understand the geometry of g by demonstrating that the conformal factor only depends on the fiber or the base, leading to two possible cases. Finally, we prove the global structure of g, arguing that both cases may not occur on a connected manifold. In Sect. 4 we give a variety of examples that demonstrate the sharpness of the classification theorems. Section 5 takes up the case in which a GQE manifold admits a structure-preserving conformal field, and Sect. 6 specializes our results to gradient Ricci solitons and m-quasi-Einstein manifolds.

2 Warped Products over a One-Dimensional Base

In this preliminary section we recall the notion of warped products over a one-dimensional base and their characterization as the metrics that support a gradient conformal field. This result has a long history: the local version goes back to Brinkmann [2], and the global version was established in full generality in the Riemannian case by Tashiro [31]. Tashiro's work generalized a well-known characterization of the sphere due to Obata [26].

We require slightly non-standard versions of these results where our metric is not complete, but is conformal to a complete metric by a conformal change of a certain form. In Lemmas 2.9 and 2.11, we establish that Tashiro's proof can be extended to give a global warped product structure in these cases, a necessary step in our eventual proof of Theorem 1.2.

Definition 2.1 A *warped product over a one-dimensional base* is a smooth manifold isometric to one of the following.

(I)

$$\left(I \times N, h = dt^2 + v(t)^2 g_N\right),$$

where I is an open interval (possibly infinite), $v : I \to \mathbb{R}$ is smooth and positive, and (N, g_N) is a Riemannian manifold;

(II)

$$\left(B_R(0) \subset \mathbb{R}^n, h = dt^2 + v(t)^2 g_{S^{n-1}}\right),$$

where $B_R(0)$ is an open ball about the origin of radius $R \in (0, \infty]$, $v : [0, R) \to \mathbb{R}$ is smooth, positive for $t > 0$, with $v(0) = 0$, and $g_{S^{n-1}}$ is a round spherical metric; or

(III)

$$\left(S^n, dt^2 + v(t)^2 g_{S^{n-1}}\right),$$

where $v : [0, R] \to \mathbb{R}$ is smooth, positive for $0 < t < R$, with $v(0) = v(R) = 0$.

Remark 2.2 In case (I) g is complete if and only if g_N is complete and $I = \mathbb{R}$. Case (II) metrics are complete if and only if $I = [0, \infty)$, and are rotationally symmetric metrics on $\mathbb{R}^n$. Case (III) metrics are rotationally symmetric metrics on S^n. In cases (II) and (III) smoothness of the metric implies further boundary conditions on the derivatives of v (see [28], for example).

An important property of these spaces is they always support a gradient conformal vector field.

Proposition 2.3 *For an n-dimensional warped product metric over a one-dimensional base*:

$$h = dt^2 + v(t)^2 g_N,$$

any anti-derivative $u(t)$ of $v(t)$ satisfies the equation

$$\frac{1}{2} L_{\nabla u} h = \mathrm{Hess}\, u = \frac{\Delta u}{n} h, \tag{2.1}$$

where ∇, Hess, *and* Δ *are the gradient*, *Hessian*, *and Laplacian with respect to h.*

The fundamental fact we will exploit is that the converse is also true. For the full proof and history of this result we refer the reader to Lemma 3.6 and Proposition 3.8 of [22], which also include pseudo-Riemannian versions and additional references. See also [9].

Lemma 2.4 *Suppose that a non-constant function u on a Riemannian manifold (M, h) satisfies* (2.1). *Then the critical points of u are non-degenerate and isolated. Fix $p \in M$.*

(1) *If $|\nabla u(p)| \neq 0$, then in a neighborhood U of p, g is isometric to a warped product over a one-dimensional base of type* (*I*):

$$(U, g) \cong \left(I \times N, dt^2 + u'(t)^2 g_N\right),$$

where $u = u(t)$, $u'(t) \neq 0$, and (N, g_N) is some Riemannian $(n-1)$-manifold independent of t. If x denotes coordinates on N, we say (t, x) give rectangular *coordinates on U.*

(2) *If* $|\nabla u(p)| = 0$ *then there is a neighborhood* U *of* p *on which* g *is isometric to a warped product over a one-dimensional base of type* (*II*), *and* u *is a function of only the distance* t *to* p:

$$(U, g) \cong \left(B_R(0), dt^2 + \left(\frac{u'(t)}{u''(0)} \right)^2 g_{S^{n-1}} \right),$$

where $u'(t) \neq 0$ *for* $t > 0$. *If* x *denotes coordinates on* S^{n-1}, *we say* (t, x) *give* polar *coordinates on* U.

Later, in case (2) we rescale $g_{S^{n-1}}$ without further comment to absorb $u''(0)$. Since aspects of the proof will be important for our next two results, we include a proof of (1) for completeness. We will find the following definitions of Tashiro useful.

Definition 2.5 Let u be a solution to (2.1). A *u-component* is a connected component of a non-degenerate level set of u. A *u-geodesic* is a geodesic of h that is parallel to ∇u wherever $\nabla u \neq 0$.

Proof of Lemma 2.4, (1) Let p be a point with $\nabla u(p) \neq 0$ and let L be the u-component containing p. There is a neighborhood U of p such that $\nabla u \neq 0$ on U, U is diffeomorphic to $(-\varepsilon, \varepsilon) \times N$, where $N \subset L$, and U has coordinates (t, x), where $\frac{\partial}{\partial t} = \frac{\nabla u}{|\nabla u|}$ and x denotes coordinates for N. We also choose N to be connected.

For X orthogonal to ∇u, (2.1) implies that

$$D_X |\nabla u|^2 = 2 \operatorname{Hess} u(X, \nabla u) = 0,$$

so $|\nabla u|$ is a function of t and $\nabla u = \psi(t) \frac{\partial}{\partial t}$. This in turn implies that $u = u(t)$ and $\psi(t) = u'(t)$. Then we have

$$h\left(\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}, X \right) = \frac{1}{u'(t)} \operatorname{Hess} u\left(\frac{\partial}{\partial t}, X \right) = 0,$$

which shows that the curves $t \mapsto (t, x)$ are the u-geodesics in U.

This establishes that the metric is of the form $h = dt^2 + g_t$, where g_t is a one-parameter family of metrics on N. It also implies that

$$\operatorname{Hess} u\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = h\left(\nabla_{\frac{\partial}{\partial t}} \left(u'(t) \frac{\partial}{\partial t} \right), \frac{\partial}{\partial t} \right) = u''(t).$$

By (2.1), we now see that $\Delta u = n u''(t)$. Using (2.1) again we obtain for X, Y orthogonal to ∇u,

$$(L_{\frac{\partial}{\partial t}} h)(X, Y) = 2 \frac{u''}{u'} h(X, Y),$$

which implies that $g_t(X, Y) = u'(t)^2 g_N(X, Y)$ for some fixed metric g_N on N. $\square$

Remark 2.6 The fact that $|\nabla u|$ is a function of t in the proof shows that we can choose U to be a neighborhood of L, even when L is non-compact: if $\nabla u \neq 0$ on $\{t_0\} \times N$, then $\nabla u \neq 0$ and has constant length on the whole leaf $\{t_0\} \times L$. In particular, we see that, in a neighborhood of a point with $\nabla u(p) \neq 0$, the sets $\{t\} \times N$ are the u-components and that the u-geodesics are the geodesics in the t direction. In the case where $\nabla u(p) = 0$ we have that all of the geodesics beginning at p are u-geodesics and the metric spheres around p are u-components.

Tashiro's theorem is the following global version of this result: if h is complete and supports a non-constant function satisfying (2.1), then h is globally a warped product over a one-dimensional base (cf. Theorem 5.4 of [27]). We show that Tashiro's arguments can be used to also prove global theorems for (possibly incomplete) metrics that are conformal to a complete metric in a certain way. First we need a definition.

Definition 2.7 Let u be a non-constant solution to (2.1) on (M, h), let f be a smooth function on M, and let $U \subset M$. We say $f = f(t)$ on U if ∇f is parallel to ∇u on U, and $f = f(x)$ on U if ∇f is orthogonal to ∇u on U.

Remark 2.8 From Lemma 2.4, around every point p, the metric h can be written on some neighborhood U of p as

$$dt^2 + u'(t)^2 g_N$$

with (polar or rectangular) coordinates (t, x). Then $f = f(t)$ in the above definition if and only if f is a function of only the t-coordinate on U; similarly, $f = f(x)$ as above if and only if f is independent of t on U.

Lemma 2.9 *Suppose that a non-constant function u on a Riemannian manifold (M, h) satisfies (2.1) and suppose that f is a function such that $f = f(t)$ on M and $(M, e^{\frac{2f}{n-2}} h)$ is complete. Then (M, h) is (globally) a one-dimensional warped product, with complete fiber metric g_N.*

Remark 2.10 The normalization of the conformal factor $e^{\frac{2f}{n-2}}$ is not important, but is used to be consistent with later notation.

Proof Set $g = e^{\frac{2f}{n-2}} h$. Let N be a u-component, with induced metric g_N from h. Applying Lemma 2.4 to every point of N shows that $g|_{TN}$ and $h|_{TN}$ are homothetic (since $f = f(t)$). Since N is a closed subset of the complete manifold (M, g), it follows that (N, g_N) is complete.

Let J be the largest open interval of regular values of u that contains $u(N)$, and let $U \subset M$ be the connected component of $u^{-1}(J)$ that contains N. Let $q \in N$ and let γ_q be the u-geodesic with respect to h through q. Since $f = f(t)$, γ_q is, up to reparameterization, also a geodesic for g. In particular, since g is complete, such curves are well defined until they possibly leave U. Moreover, since $u = u(t)$, they all leave U (if at all) at the same parameter value of t.

As in the proof of Lemma 2.4, it follows that U is diffeomorphic to $I \times N$, where I is an open interval, and that in the coordinates induced by this diffeomorphism,

$$h = dt^2 + u'(t)^2 g_N,$$

where t is the signed h-distance to g_N. Say $I=(a,b)$ (where $a,b\in[-\infty,\infty]$), and define the change of variables

$$s(t)=\int_0^t e^{\frac{f(r)}{n-2}}\,dr.$$

Using the new coordinate s, we have that on U,

$$g=ds^2+u'(s)^2e^{\frac{4f(s)}{n-2}}g_N.$$

Define

$$c=\lim_{t\to a+}s(t),\qquad d=\lim_{t\to b-}s(t).$$

Now we can just imitate Tashiro's proof, analyzing three possible cases.

If $c=-\infty$ and $d=+\infty$, the restriction of g to the open subset U defines a complete metric, and so $(U,g|_U)=(M,g)$. In particular, g is a globally warped product with one-dimensional base of type (I). Since $f=f(t)$, the same goes for h.

Next, suppose $c=-\infty$ but d is finite (or vice versa). Consider a geodesic $\gamma(s)$ with respect to g, orthogonal to N with increasing s. By completeness, γ may be extended to $\mathbb{R}$, and so we conclude $q=\lim_{s\to d-}\gamma(s)$ is a critical point of u (or else J was not maximal as chosen). By Lemma 2.4, h admits polar coordinates (t_1,x) about p with warping factor u' and fiber S^{n-1}. Since the coordinates t and t_1 are both given by level sets of u, these coordinate neighborhoods can be combined to show that h and g are warped product metrics with one-dimensional base of type (II).

Finally, consider the case in which c and d are both finite. A similar argument shows that u has critical points at $s=c$ and $s=d$, and we conclude that h is a warped product with one-dimensional base of type (III). □

From these arguments we also see the following in the case $f=f(x)$.

Lemma 2.11 *Suppose that a non-constant function u on a Riemannian manifold (M,h) satisfies (2.1) and has no critical points. Suppose that f is a function such that $f=f(x)$ on M and $(M,e^{\frac{2f}{n-2}}h)$ is complete. Then (M,h) is (globally) a one-dimensional warped product of type (I):*

$$(M,h)=\left(\mathbb{R}\times N,dt^2+u'(t)^2g_N\right).$$

Remark 2.12 In this case, g_N is not necessarily complete.

Proof This result follows from similar arguments to Lemmas 2.4 and 2.9 once we show that the u-geodesics with respect to h exist for all time. Let $\gamma(t)$ be a u-geodesic with respect to h defined for $0\le t<t_0$. Let $t_i\nearrow t_0$, so that $\{\gamma(t_i)\}$ is Cauchy in (M,h). The sequence is also Cauchy in (M,g) since $g=e^{\frac{2f}{n-2}}h$ and f is constant along $\gamma(t)$. By completeness, $\{\gamma(t_i)\}$ converges with respect to g to some $q\in M$. By considering the u-geodesics in a neighborhood of q, we see that γ can be extended past t_0. □

Remark 2.13 From the proof we can see that only the completeness of the u-geodesics in g is necessary for this last result.

3 Conformal Diffeomorphisms Preserving the GQE Structure

In this section we prove Theorem 1.2, giving the classification of GQE metrics admitting a conformal diffeomorphism preserving the GQE structure. The first step is to give a convenient conformal interpretation of the GQE equation (1.1), used previously by Catino [7] and Kotschwar [20].

Proposition 3.1 *A Riemannian manifold (M, g) of dimension $n \geq 3$ satisfies the GQE equation (1.1) with functions f, α, and λ if and only if there is a conformally related metric h that satisfies*

$$\mathrm{Ric}_h = \left(\frac{1}{n-2} - \alpha\right) df \otimes df + Qh \tag{3.1}$$

for some function Q, where Ric_h is the Ricci curvature of h.

Proof Set $h = e^{\frac{-2f}{n-2}} g$. The Ricci curvatures of h and g are related by

$$\mathrm{Ric}_h = \mathrm{Ric}_g + \mathrm{Hess}_g f + \frac{1}{n-2} df \otimes df + \frac{1}{n-2}\left(\Delta_g f - |\nabla f|_g^2\right) g. \tag{3.2}$$

Thus we see that h satisfies (3.1) if and only if $\mathrm{Ric}_g + \mathrm{Hess}_g f + \alpha df \otimes df = \lambda g$, where

$$Q = \frac{1}{n-2}\left(\Delta_g f - |\nabla f|_g^2 + (n-2)\lambda\right) e^{\frac{2f}{n-2}}. \tag{3.3}$$

□

Remark 3.2 h is generally incomplete even if g is complete.

Remark 3.3 It follows that a Riemannian metric is conformal to an Einstein metric if and only if it admits a GQE structure with $\alpha \equiv \frac{1}{n-2}$.

Example 3.4 A warped product over a one-dimensional base

$$h = dt^2 + v(t)^2 g_N$$

has Ricci curvature

$$\mathrm{Ric}_h = -(n-1)\frac{v''}{v} dt^2 + \mathrm{Ric}_{g_N} - \left(vv'' + (n-2)\left(v'\right)^2\right) g_N. \tag{3.4}$$

Assume $\mathrm{Ric}_{g_N} = \mu g_N$ for a constant μ. Then (3.1) is satisfied with $f = f(t)$ if and only if

$$\begin{aligned}\left(\frac{1}{n-2} - \alpha\right) f'(t)^2 &= \mathrm{Ric}_h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) - \mathrm{Ric}_h(X, X) \\ &= \frac{(n-2)((v')^2 - vv'') - \mu}{v^2},\end{aligned} \tag{3.5}$$

where X is a field perpendicular to $\frac{\partial}{\partial t}$ such that $h(X, X) = 1$. Choose a function $\alpha(t)$ so that (3.5) defines a function $f(t)$. Then h and f satisfy (3.1) for some Q. Consequently, $g = e^{\frac{2f}{n-2}} h$ is a GQE metric.

Example 3.5 As an alternative to the above, let (M, g) be a warped product over a one-dimensional base, $g = ds^2 + v(s)^2 g_N$, of type (I) with g_N Einstein (with Einstein constant μ), or else of type (II) or (III) (in which the Einstein constant of $g_{S^{n-1}}$ is $(n-1)$). Then (M, g) is automatically a Ricci almost soliton, where the potential $f = f(s)$ can be found from the ODE

$$\left(\frac{f'}{v}\right)' = \frac{\mu + (n-2)(vv'' - (v')^2)}{v^3},$$

and $\lambda = \lambda(s)$ is given by

$$\lambda = f'' - (n-1)\frac{v''}{v}.$$

3.1 Local Form of h

We first prove a local classification for the conformally rescaled metric h.

Lemma 3.6 *Let $(M_1, g_1, f_1, \alpha_1, \lambda_1)$ and $(M_2, g_2, f_2, \alpha_2, \lambda_2)$ be GQE manifolds admitting a non-homothetic conformal diffeomorphism ϕ preserving the GQE structure. Then every $p \in M_1$ is contained in a neighborhood U on which $h_1 = e^{-\frac{2f_1}{n-2}} g_1$ is a warped product over a one-dimensional base of type (I) or (II):*

$$h_1 = dt^2 + u'(t)^2 g_N$$

for an appropriate function $u(t)$.

Remark 3.7 If $\alpha_1 = \alpha_2 = \frac{1}{n-2}$ or f_1 and f_2 are constant, this result recovers Brinkmann's original result for Einstein manifolds.

Proof Let $h_i = e^{-\frac{2f_i}{n-2}} g_i$ be the corresponding conformally rescaled metrics. Then $\phi^* g_2 = w^{-2} g_1$ if and only if $\phi^*(h_2) = u^{-2} h_1$, where

$$u^{-2} = w^{-2} e^{\frac{2f_1}{n-2}} e^{-\frac{2\phi^* f_2}{n-2}} = w^{-2} e^{-\frac{2C}{n-2}}$$

for the constant $C = \phi^* f_2 - f_1$. In particular, since w is non-constant, u is non-constant as well. We have from Proposition 3.1 that $\mathrm{Ric}_{h_i} = (\frac{1}{n-2} - \alpha_i) df_i \otimes df_i + Q_i h_i$ for $i = 1, 2$. Since ϕ preserves the GQE structure, we have

$$\phi^*\left(\left(\frac{1}{n-2} - \alpha_2\right) df_2 \otimes df_2\right) = \left(\frac{1}{n-2} - \alpha_1\right) df_1 \otimes df_1,$$

and so

$$\mathrm{Ric}_{\phi^*(h_2)} - \mathrm{Ric}_{h_1} = \left(\phi^* Q_2\right)\phi^* h_2 - Q_1 h_1 = \left(\left(\phi^* Q_2\right) u^{-2} - Q_1\right) h_1.$$

Thus the difference of the Ricci tensors is pointwise proportional to h_1. On the other hand, by the formula for the conformal change $h_1 \to u^{-2}h_1$, we have

$$\mathrm{Ric}_{\phi^*(h_2)} - \mathrm{Ric}_{h_1} = (n-2)\frac{\mathrm{Hess}_{h_1} u}{u} + \left(u^{-1}\Delta_{h_1} u - (n-1)|\nabla u|^2_{h_1} u^{-2}\right)h_1.$$

Putting these equations together we conclude that $\mathrm{Hess}_{h_1} u$ is pointwise proportional to h_1. Taking the trace, we see that (2.1) is satisfied by u with respect to h_1. From Lemma 2.4, we deduce the local warped product structure of h_1. □

3.2 Local Form of g

Now with a local classification of the metric $h_1 = e^{-\frac{2f_1}{n-2}} g_1$, we pass to a local classification of g_1 by understanding the local behavior of f_1.

Lemma 3.8 *Under the hypotheses of Lemma 3.6, let p be a point in M_1 with $\alpha_1(p) \neq \frac{1}{n-2}$, and set $h = h_1$ and $f = f_1$, and $\alpha = \alpha_1$.*

(1) *If p is a critical point of u, then h is a type (II) warped product*

$$h = dt^2 + u'(t)^2 g_{S^{n-1}}$$

in a polar coordinate neighborhood U of p and $f = f(t)$ on U.

(2) *If p is not a critical point of u, then h is a type (I) warped product*

$$h = dt^2 + u'(t)^2 g_N$$

in a rectangular coordinate neighborhood U of p, and either

(a) *$f = f(t)$ on U and (if $n \geq 4$ or α is constant) g_N is Einstein, or*

(b) *$f = f(x)$ on U and*

$$\mathrm{Ric}_{g_N} = \left(\frac{1}{n-2} - \alpha\right) df \otimes df + P g_N, \tag{3.6}$$

where P is a constant and $\alpha = \alpha(x)$. Moreover, Q, defined in Proposition 3.1, is constant and $u''' = \frac{-Q}{n-1}u'$.

Finally, case (2b) does not occur if $n = 3$.

In particular, in the $f = f(x)$ case, the metric $e^{\frac{2f}{n-3}} g_N$ is a GQE $(n-1)$-manifold with potential f, with α shifted by $\frac{1}{n-3} - \frac{1}{n-2}$ (by Proposition 3.1).

Proof By the previous lemma, in a neighborhood U of p,

$$h = dt^2 + u'(t)^2 g_N,$$

where (t, x) are either polar or rectangular coordinates and the metric g_N is independent of t. Letting $v(t) = u'(t)$, we find the Ricci curvature of h from (3.4). For X, Y

tangent to N,

$$\begin{aligned}
&\mathrm{Ric}_h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -(n-1)\frac{v''}{v} \\
&\mathrm{Ric}_h\left(\frac{\partial}{\partial t}, X\right) = 0 \\
&\mathrm{Ric}_h(X, Y) = \mathrm{Ric}_{g_N}(X, Y) - \left(\frac{v''}{v} + (n-2)\frac{(v')^2}{v^2}\right)h(X, Y).
\end{aligned} \tag{3.7}$$

In this proof, we frequently identify $\{t\} \times N$ with N.

We begin with some observations. First, from (3.1) we see that Ric_h has at most two distinct eigenvalues at each point. If there are two distinct eigenvalues, the orthogonal eigenspaces are of dimension 1 and $n-1$. Second, ∇f is an eigenvector field for the one-dimensional eigenspace of Ric_h wherever it does not vanish. Third, from (3.7), $\frac{\partial}{\partial t}$ is an eigenvector field for Ric_h.

Fix $p \in M$, and let U be a coordinate neighborhood as in Lemma 3.6 (shrunken if necessary so that $\alpha \neq \frac{1}{n-2}$ on U).

Case A If p is a critical point of u then we have polar coordinates (t, x), and $g_N = g_{S^{n-1}}$. We then have

$$\mathrm{Ric}_h(X, Y) = \left(\frac{\mu - (n-2)(v')^2}{v^2} - \frac{v''}{v}\right)h(X, Y),$$

where μ is the Einstein constant of g_{S^n}. This shows that the $(n-1)$-dimensional space $T_qN \subset T_qM$ is contained in an eigenspace of Ric_h for every $q \in U$. Therefore, at any point in U for which $\nabla f \neq 0$, we have that ∇f and $\frac{\partial}{\partial t}$ both span the one-dimensional eigenspace of Ric_h so that ∇f is parallel to $\frac{\partial}{\partial t}$. Then $f = f(t)$ on U.

From now on, we assume p is not a critical point of u, so that we have rectangular coordinates (t, x) on U. Without loss of generality we also assume that $p = (0, x_0)$ in these coordinates.

Case B Suppose p is not a critical point of u and $\mathrm{Ric}_{g_N} = \mu g_N$ at all points in a neighborhood $V \subset N$ containing x_0 for some function $\mu \in C^\infty(V)$. (This assumption is always satisfied when $n = 3$ and by Schur's lemma μ is constant if we are in this case and $n > 3$). Then the exact same argument as in Case A, which only used $\mathrm{Ric}_{g_N} = \mu g_N$, shows that $f = f(t)$ on U. Moreover, if α is constant, then (3.1) and (3.7) show that $Q = Q(t)$ and consequently that μ is independent of x and therefore constant.

We are now left with the case that p is not a critical point of u and $\mathrm{Ric}_{g_N} \neq \mu g_N$ in any neighborhood of x_0. There are then two cases, depending on whether the condition is true at x_0 or not.

Case C Suppose p is not a critical point of u and Ric_{g_N} is not proportional to g_N at x_0.

Then Ric_h is not proportional to h, so $\nabla f(p) \neq 0$. Shrink U if necessary so that $\nabla f \neq 0$ on U. ∇f is an eigenvector field of Ric_h with corresponding eigenvalue of multiplicity one. It cannot be parallel to $\frac{\partial}{\partial t}$ and therefore must be orthogonal to $\frac{\partial}{\partial t}$ on U. This shows that $f = f(x)$ on U. Setting our two expressions for Ric_h equal yields

$$\begin{aligned} -(n-1)\frac{v''}{v} &= Q \\ \mathrm{Ric}_{g_N} &= \left(\frac{1}{n-2} - \alpha\right) df \otimes df + \underbrace{\left(Qv^2 + vv'' + (n-2)\left(v'\right)^2\right)}_{P} g_N. \end{aligned} \tag{3.8}$$

The first equation implies that $Q = Q(t)$; it follows then from the second that the coefficient P on g_N is constant, and consequently $\alpha = \alpha(x)$. Eliminating Q in the equations gives

$$vv'' - \left(v'\right)^2 = -\frac{P}{n-2}.$$

Any solution to this equation must solve $v'' = kv$ for some constant k and therefore Q must also be constant. We then find for later reference that

$$\left(v'\right)^2 + \frac{Q}{n-1}v^2 = \frac{P}{n-2}. \tag{3.9}$$

Case D p is not a critical point of u and Ric_{g_N} is proportional to g_N at x_0, but is not proportional in any neighborhood of x_0.

Consider a sequence of points x_i in N converging to x_0 such that Ric_{g_N} is not proportional to g_N at x_i. Then $\nabla f(t, x_i)$ must be tangent to N by case C. Therefore, by continuity and (3.8), we must have $df = 0$ at the points (t, x_0) $t \in I$. If $df = 0$ in a neighborhood of p, then f is constant and the lemma is clearly true by simply shrinking U to be the neighborhood where f is constant.

Consider a connected component W of the nonempty, open set $\{df \neq 0\} \cap U$ with $p \in \partial W$. Since $df \neq 0$ on W, we have by cases A–C that either $f = f(t)$ or $f = f(x)$ on W. By way of contradiction, suppose $f = f(t)$ on W. Then W is a set of the form $(a, b) \times V$, where V is an open subset of N. By the previous paragraph, f is constant along the curve $t \mapsto (t, x_0)$. If $x_0 \in V$, this shows that df vanishes in W, a contradiction. If $x_0 \in \partial V$, the same argument applies by continuity. Therefore, we have $f = f(x)$ in a neighborhood of p, so we may follow the argument of case C. □

3.3 Global Form of g

The previous lemma splits M into two sets: the points where $f = f(t)$ and the points where $f = f(x)$. We now rule out the possibility that both cases occur.

Lemma 3.9 *If $\alpha \neq \frac{1}{n-2}$, the cases $f = f(x)$ and $f = f(t)$ may not both occur on the same connected manifold M, unless f is constant. Moreover, if f is non-constant and $f = f(x)$ occurs, then u has no critical points.*

Proof Define A to be the set of points $p \in M$ that are either critical points of u or regular points p that satisfy the property

> ∇f is everywhere orthogonal to L and $|\nabla f|$ is constant along L, where L is the u-component containing p,

which we denote by $(*)$. As usual, ∇ is the gradient with respect to h.

We show A is open. Let $p \in A$. First, if p is a critical point, then by the previous lemma, we have polar coordinates around p with $f = f(t)$. On this coordinate neighborhood, $(*)$ clearly holds at every point besides p. Otherwise, p is a regular point; let L be the u-component containing it. If $\nabla f(p) \neq 0$, then we are in the $f = f(t)$ case on a neighborhood U constructed in the previous lemma. It is readily seen that $(*)$ holds on U.

If $\nabla f(p) = 0$, then by $(*)$, ∇f vanishes on L. Then on L, the Ricci curvature of L is proportional to the metric on L, by (3.1) and (3.4). Let U be a coordinate neighborhood of p as in the previous lemma, so that

$$h = dt^2 + u'(t)^2 g_N$$

on U, where $N \subset L$. By case B of the proof of the previous lemma, $f = f(t)$ on U, and so $(*)$ holds on U.

Next, we show A is closed. Let $\{p_i\}$ be a sequence in A converging to $p \in M$. Since A is open and the critical points of u are isolated, we may assume without loss of generality that each p_i is a regular point of u. If p is a critical point of u, we are done. Otherwise, let L be the u-component containing p, and similarly L_i for p_i. It is now clear from the definition that $(*)$ holds on L, since it holds on each L_i.

Thus, either A is empty, or $A = M$. If $f = f(x)$ and is non-constant on some open set, then ∇f is tangent to a u-component, and so A is empty. In particular, u has no critical points. If, in addition, $f = f(t)$ and is non-constant on an open set, then A is non-empty, a contradiction. □

Now we complete the proof of the local and global classifications stated in the Introduction.

Proof of Theorem 1.2 From Lemmas 3.6 and 3.8, $g_1 = e^{\frac{2f_1}{n-2}} h_1$ is locally either of the form (1.2) or (1.3), (since in the $f_1 = f_1(t)$ case we may do a change of variables $ds = e^{\frac{f_1(t)}{n-2}} dt$). Moreover, if g_1 is complete, then we have a global structure of the form (1.2) or (1.3) by Lemmas 2.9 and 2.11. In the $f_1 = f_1(x)$ case, (3.6) is satisfied, which implies by Proposition 3.1 that $e^{\frac{2f_1}{n-3}} g_N$ is GQE with potential f_1.

Next, we prove that h_2 also has a warped product structure. Since

$$h_1 = dt^2 + u'(t)^2 g_N,$$

the metric h_2 satisfies

$$\phi^*(h_2) = u^{-2}(t) dt^2 + \left(u' u^{-1}\right)^2 g_N.$$

Defining $dr = u^{-1}(t)dt$ we then obtain that

$$h_2 = dr^2 + \left(\frac{d}{dr}(-u^{-1})\right)^2 g_N$$

up to isometry. Note that since $\phi^* df_2 = df_1$, if $f_1 = f_1(t)$ then $f_2 = f_2(t)$ (and similarly if $f_1 = f_1(x)$). Thus, g_2 has the form stated in the theorem. $\square$

We also prove the global result for compact manifolds.

Proof of Theorem 1.6 By compactness and Lemma 2.9 (or alternatively by Tashiro's theorem), h_1 is a warped product with one-dimensional base of type (III), and is therefore a rotationally symmetric metric on a sphere. Such metrics are conformal to a round metric, so certainly (M_1, g_1) is conformally diffeomorphic to a round sphere, and the same goes from (M_2, g_2).

Suppose $\alpha_1 \neq \frac{1}{n-2}$. Since u has a critical point by compactness, we are in the case $f_1 = f_1(t)$ by Lemma 3.9. It then follows that (M_1, g_1) and (M_2, g_2) are rotationally symmetric metrics on the sphere. $\square$

4 Examples

In the next two subsections we construct examples of generalized quasi-Einstein manifolds that are warped products over a one-dimensional base, both in the $f = f(t)$ and $f = f(x)$ cases. By Proposition 3.1, it suffices to construct metrics h and functions f satisfying (3.1). In the third subsection we further show that all of these examples admit one-parameter families of local conformal changes that preserve the GQE structure. In many cases these conformal changes are global.

4.1 The $f(t)$ Case

We start with an arbitrary Riemannian manifold (U, h) of dimension $n \geq 3$ of the form

$$U = (a, b) \times N$$
$$h = dt^2 + u'(t)^2 g_N$$

for some $u(t)$ with $u'(t) > 0$ on (a, b). Assume that N is an Einstein metric, $\mathrm{Ric}_{g_N} = \mu g_N$ and that $\alpha : (a, b) \to \mathbb{R}$ is also a smooth function of t, such that $\alpha(t) \neq \frac{1}{n-2}$ for any t. From Example 3.4 we see that there is a metric of the form $g = e^{\frac{2f}{n-2}} h$ for some function $f = f(t)$ if u satisfies the differential inequality

$$\frac{1}{\frac{1}{n-2} - \alpha}\left(-(n-2)\frac{u'''}{u'} - \frac{\mu - (n-2)(u'')^2}{(u')^2}\right) \geq 0 \tag{4.1}$$

on (a, b).

Remark 4.1 This shows that, given a warping function $u > 0$ such that the derivatives of u are bounded on (a, b), and any $\alpha \neq \frac{1}{n-2}$, there is a choice of Einstein metric g_N so that the metric admits a GQE structure on (a, b).

From formula (3.5) an equivalent way to state this result is as follows.

Proposition 4.2 *Let (U, h), (N, g_N), and α be as above. Then $(U, e^{\frac{2f}{n-2}}h, f, \alpha, \lambda)$ is a GQE manifold for some λ and f if and only if*

(1) $\mathrm{Ric}_h(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \geq \mathrm{Ric}_h(X, X)$ *on* (a, b), *when* $\alpha < \frac{1}{n-2}$, *and*
(2) $\mathrm{Ric}_h(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \leq \mathrm{Ric}_h(X, X)$ *on* (a, b), *when* $\alpha > \frac{1}{n-2}$.

Example 4.3 We consider the concrete example

$$h = dt^2 + e^{2t} g_N$$

so that

$$\left(\frac{1}{n-2} - \alpha\right)\left(f'\right)^2 = -\mu e^{-2t}$$

by (3.5). If g_N has Einstein constant $\mu < 0$, we may choose $\alpha = 0$, for instance, so that f equals

$$f(t) = \pm\sqrt{-\mu(n-2)}e^{-t} + C.$$

If $\mu > 0$ we can choose $\alpha = \frac{2}{n-2}$, for instance, and so $f(t) = \pm\sqrt{\mu(n-2)}e^{-t} + C$. Finally, if $\mu = 0$, then h is Einstein.

This construction also works in polar coordinate neighborhoods.

Proposition 4.4 *Suppose that*

$$U = B_R(0)$$
$$h = dt^2 + u'(t)^2 g_{S^{n-1}}$$

is a polar coordinate neighborhood of a smooth metric h, and suppose that $\alpha = \alpha(t)$ is a smooth function on U, never equal to $\frac{1}{n-2}$. Then $(U, e^{\frac{2f}{n-2}}h, f, \alpha, \lambda)$ is a GQE manifold for some λ and f if and only if (4.1) holds on $[0, R)$ with $\mu = n-2$.

Proof Define $f(t)$ to solve (3.5) on $(0, R)$ (where $v = u'$). We know that $u' \to 0$ as $t \to 0$. From the smoothness of the metric, it also follows that

$$\mathrm{Ric}_h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \longrightarrow \mathrm{Ric}_h(X, X) \quad \text{as } t \to 0.$$

Since $\alpha \nrightarrow \frac{1}{n-2}$ as $t \to 0$, f extends to a smooth function on U with a critical point at $t = 0$. $\square$

If we do not prescribe α, these propositions give the following corollary.

Corollary 4.5 *Let (U, h) be any warped product over a one-dimensional base with fiber metric g_N Einstein. If h has non-constant curvature at almost every point, then there are functions $\alpha = \alpha(t)$, $f = f(t)$, and $\lambda = \lambda(t)$ such that $(U, e^{\frac{2f}{n-2}} h, f, \alpha, \lambda)$ is a* complete *generalized quasi-Einstein structure.*

Proof First define $\alpha(t)$ so that it has the following properties:

- $\alpha(t) < \frac{1}{n-2}$ at points in (a, b) where, $\mathrm{Ric}_h(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \geq \mathrm{Ric}_h(X, X)$,
- $\alpha(t) > \frac{1}{n-2}$ at points in (a, b) where, $\mathrm{Ric}_h(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \leq \mathrm{Ric}_h(X, X)$, and
- $\alpha(t) = \frac{1}{n-2}$ at points in (a, b) where, $\mathrm{Ric}_h(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \mathrm{Ric}_h(X, X)$.

Defining $f(t)$ via (3.5), we have a GQE manifold structure on $(a, b) \times N$. By the hypothesis on curvature, f' vanishes only on a set of measure zero. If h is a type (I) warped product, this gives a GQE structure on U. Then we can also choose $\alpha(t) \to \frac{1}{n-2}$ fast enough as t limits to a and b so that we can make $f'(t)$ blow up at the endpoints so that $s(t) = \int_0^t e^{\frac{f}{n-2}} dt$ limits to $-\infty$ as $t \to a^+$ and limits to ∞ as $t \to b^-$. This implies that $e^{\frac{2f}{n-2}} h$ is a complete metric of type (I).

When we have a type (II) or (III) warped product, we also must modify α so that f extends to a smooth function in the polar coordinate neighborhood. To do this, choose α such that

$$\mathrm{Ric}_h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) - \mathrm{Ric}_h(X, X) = o\left(\frac{1}{n-2} - \alpha\right)$$

as $t \to a$ or b, and so that the same condition holds for all derivatives.

In the type (III) case this gives us the desired complete metric. In the type (II) case we obtain a complete metric by controlling the asymptotics of α in the same way as in the type (I) case. □

Remark 4.6 As this construction shows, the function $\alpha(t)$ is not unique.

Remark 4.7 If $n = 3$, it is possible to have solutions in the $f = f(t)$ case such that α is a function of both t and x. By Schur's lemma applied to g_N, these examples are not possible in dimension above three.

For example, let Σ be any surface with Gauss curvature $\mu(x)$. The metric

$$h = dt^2 + \cosh^2(t) g_\Sigma$$

with $f(t) = \int_0^t \frac{dr}{\sqrt{\cosh(r)}}$ and $\alpha = 1 + \frac{1+\mu(x)}{\cosh(t)}$ satisfies (3.1). Moreover, f is bounded, so $g = e^{2f} h$ is complete, provided Σ is chosen to be complete.

4.2 The $f(x)$ Case

In this section we construct non-Einstein examples in the $f = f(x)$ case with dimension $n \geq 4$. The approach is to begin with a metric g_N on an $(n-1)$-manifold N

satisfying

$$\mathrm{Ric}_{g_N} = \left(\frac{1}{n-2} - \alpha\right) df \otimes df + P g_N \tag{4.2}$$

for some function f on N and constants $\alpha \neq \frac{1}{n-2}$ and P.

Example 4.8 The simplest example of a metric solving (4.2) is a product metric $g_N = dy^2 + g_F$ where g_F is an Einstein $(n-2)$-manifold with Einstein constant P. If $P = 0$, then we obtain a Ricci-flat metric so f is constant. However, if $P < 0$ we may choose $\alpha < \frac{1}{n-2}$ constant to obtain a solution with f a linear function of y. We also obtain solutions when $P > 0$ by letting $\alpha > \frac{1}{n-2}$ be constant.

We point out that one may obtain examples with f bounded in the case $P \neq 0$ by choosing $\alpha = \alpha(y)$ appropriately. In particular, if g_F is chosen to be complete, one may find complete GQE metrics $g = e^{\frac{2f}{n-2}} h$ in the $f = f(x)$ case, where h is given in (4.3) in the construction below.

To construct nontrivial examples in the $P = 0$ case, we take the complete metric $g_N = dy^2 + (1 + y^2) g_{S^{n-2}}$ for $n \geq 4$. The Ricci curvature of g_N is

$$\mathrm{Ric}_{g_N} = -\frac{n-2}{(1+y^2)^2} dy^2.$$

Choosing $\alpha = \frac{1}{n-2} + n - 2$ and $f(y) = \arctan(y)$ ensures that (4.2) holds with $P = 0$. Moreover, since f is bounded, $g = e^{\frac{2f}{n-2}} h$ will be complete.

Let $Q \in \mathbb{R}$ and define

$$h = dt^2 + v(t)^2 g_N, \tag{4.3}$$

where $v(t)$ is a nonzero solution to $v'' = -\frac{Q}{n-1} v$ that is nonnegative over the range of t. Direct calculation shows that

$$\mathrm{Ric}_h = \left(\frac{1}{n-2} - \alpha\right) df \otimes df + Q g_N,$$

where v solves (3.9), restated here for convenience:

$$\left(v'\right)^2 + \frac{Q}{n-1} v^2 = \frac{P}{n-2}.$$

By rescaling g_N, we normalize so that $P = n-2$, 0, or $-(n-2)$. There are six cases, up to an affine change of variable in t:

(1) If $P = n-2$, there are three cases:
 (a) $v(t) = \sin(t)$, $Q = n-1$,
 (b) $v(t) = t$, $Q = 0$, and
 (c) $v(t) = \sinh(t)$, $Q = -(n-1)$.

(2) If $P=0$, there are two cases:
 (a) $v(t)=1$, $Q=0$ and
 (b) $v(t)=e^t$, $Q=-(n-1)$.
(3) If $P=-(n-2)$, there is only one case:
 (a) $v(t)=\cosh(t)$, $Q=-(n-1)$.

4.3 Conformal Changes

For the examples constructed in the last two subsections, we show that around every point the metric admits local non-homothetic conformal changes preserving the GQE structure.

Proposition 4.9 *Let $h_1=dt^2+u'(t)^2 g_N$, and suppose that there are functions f, Q_1, and α such that*

$$\mathrm{Ric}_{h_1}=\left(\frac{1}{n-2}-\alpha\right)df\otimes df+Q_1h_1. \tag{4.4}$$

Then $h_2=u^{-2}h_1$ satisfies

$$\mathrm{Ric}_{h_2}=\left(\frac{1}{n-2}-\alpha\right)df\otimes df+Q_2h_2, \tag{4.5}$$

where $Q_2=(n-1)(\frac{Q_1}{n-1}u^2+2u''u-(u')^2)$.

Proof This essentially follows from the observation that the steps in the proof of Lemma 3.6 can be reversed. Namely, we know that, for the metric h_1, we have

$$\mathrm{Hess}_{h_1}u=\frac{\Delta_{h_1}u}{n}h_1.$$

Therefore, the formula for the change of the Ricci tensor tells us that:

$$\begin{aligned}\mathrm{Ric}_{h_2}&=\mathrm{Ric}_{h_1}+(n-2)\frac{\mathrm{Hess}_{h_1}u}{u}+\big(u^{-1}\Delta_{h_1}u-(n-1)|\nabla u|^2_{h_1}u^{-2}\big)h_1\\&=\mathrm{Ric}_{h_1}+(n-1)\left(\frac{2}{n}u^{-1}\Delta_{h_1}u-|\nabla u|^2_{h_1}u^{-2}\right)h_1\\&=\left(\frac{1}{n-2}-\alpha\right)df\otimes df+(n-1)\left(\frac{Q_1}{n-1}u^2+\frac{2}{n}u\Delta_{h_1}u-|\nabla u|^2_{h_1}\right)h_2.\end{aligned}$$

The formula then follows from $\Delta_{h_1}u=nu''$ and $|\nabla u|^2_{h_1}=(u')^2$ on h_1. □

As a corollary we obtain conformal diffeomorphisms between the generalized quasi-Einstein manifolds constructed in the previous subsections.

Corollary 4.10 *Let $(U,e^{\frac{2f}{n-2}}h,f,\alpha,\lambda_1)$ be a GQE manifold with $h_1=dt^2+u'(t)^2g_N$ for some function $u(t)$. Then there is a function λ_2 such that $(U,u^{-2}e^{\frac{2f}{n-2}}h,$ $f,\alpha,\lambda_2)$ is also a GQE manifold.*

Proof Set $h_1 = h$, $g_1 = e^{\frac{2f}{n-2}} h_1$, $h_2 = u^{-2} h_1$, and $g_2 = e^{\frac{2f}{n-2}} h_2$. From Proposition 3.1 we know that (4.4) holds, with

$$Q_1 = \frac{1}{n-2}\left(\Delta_{g_1} f - |\nabla f|^2_{g_1} + (n-2)\lambda_1\right) e^{\frac{2f}{n-2}}$$
$$= \frac{\Delta_{h_1} f}{n-2} + \lambda_1 e^{\frac{2f}{n-2}}.$$

Proposition 4.9 then implies that (4.5) holds, with

$$Q_2 = (n-1)\left(u^2 \frac{Q_1}{n-1} + 2u'' u - \left(u'\right)^2\right)$$
$$= \frac{u^2 \Delta_{h_1} f}{n-2} + \lambda_1 e^{\frac{2f}{n-2}} u^2 + (n-1)\left(2u''u - \left(u'\right)^2\right).$$

Direct computation now shows:

$$\lambda_2 = \left(\lambda_1 e^{\frac{2f}{n-2}} u^2 + uh(\nabla u, \nabla f) + (n-1)\left(2u''u - \left(u'\right)^2\right)\right) e^{\frac{-2f}{n-2}}.$$

Note that the term $uh(\nabla u, \nabla f)$ vanishes in the $f = f(x)$ case and equals $uu'f'$ in the $f = f(t)$ case. □

We can also see that h_2 is isometric to a warped product over a one-dimensional base with the same fiber as h_1; see the proof of Theorem 1.2 in Sect. 3.3.

Also note that we have, in fact, constructed a one-parameter family of conformal changes, as we can choose u to be any anti-derivative of the warping function. The next elementary example (which also appears in [22]) shows that the choice of anti-derivative does impact the behavior of the conformally changed metric.

Example 4.11 Suppose that we have the standard round metric on S^n:

$$dr^2 + \sin^2(t) g_{S^{n-1}}.$$

Then we can choose $u(t) = c - \cos(t)$. The choice $c > 1$ gives a function u which is positive everywhere on the sphere, and the conformally changed metric will also be a round sphere (of possibly different curvature). When $c \leq 1$, u will not be positive, so we do not have a global conformal change. However, when $c = 1$ we obtain stereographic projection from the sphere minus a point to Euclidean space. When $0 < c < 1$ we obtain a conformal change from a portion of the sphere to a portion of hyperbolic space, possibly rescaled.

We also note that with a rotationally symmetric metric (i.e., if u' vanishes somewhere), it is always possible to choose the conformal factor u to be positive everywhere: since $u'(t) > 0$, u is bounded from below and thus can be made positive by adding a suitable constant.

5 Conformal Fields

We prove local and global classification results for GQE manifolds admitting conformal fields. In this section we assume:

(1) $(M, g, f, \alpha, \lambda)$ is a GQE manifold with $\alpha \neq \frac{1}{n-2}$,
(2) V is a vector field on M such that $L_V g = 2\eta g$, with η non-constant (i.e., V is a non-homothetic conformal field), and
(3) V preserves the GQE structure, in the sense that $D_V f$ equals a constant c, and $D_V \alpha = 0$.

First, note that V is also a non-homothetic conformal field for $h = e^{-\frac{2f}{n-2}} g$:

$$L_V h = 2\left(\eta - \frac{c}{n-2}\right)h.$$

We define $\sigma = \eta - \frac{c}{n-2}$.

Next, we make the following observations. If ϕ_t is the local flow of V about some point, then $\phi_t^* g = w_t^{-2} g$ for a smooth family of functions w_t. The smooth family $u_t = w_t e^{\frac{C(t)}{n-2}}$ (where $\phi_t^* f = f + C(t)$) satisfies $\phi_t^* h = u_t^{-2} h$, and therefore solves (2.1) for each t with respect to $h = e^{-\frac{2f}{n-2}} g$, by the proof of Lemma 3.6. Differentiating in t, we find that η satisfies equation (2.1) on M[2] (even though the local flows of V may not be globally defined). Following the same arguments as in Sect. 3, we have:

Observation 5.1 *The non-constant function σ satisfies* (2.1), *and the local and global classification results* (*Lemmas* 3.6, 3.8, 3.9 *and Theorems* 1.2 *and* 1.6) *hold in the present case, with u replaced by σ.*

Consequently, h is of the form (locally or globally)

$$h = dt^2 + \sigma'(t)^2 g_N \tag{5.1}$$

for $t \in I$. We similarly define $f = f(t)$ (resp., $f = f(x)$) to mean ∇f is parallel (resp. orthogonal) to $\nabla\sigma$.

We are therefore led to study conformal fields on a warped product over a one-dimensional base. We fix notation for V by writing

$$V = v_0(t, x)\frac{\partial}{\partial t} + V_t,$$

where v_0 is some function on $I \times N$, and V_t is the projection of V onto the factor $\{t\} \times N$. Some general facts regarding this case are collected in the statement below, which follows immediately from Proposition A.1 in Appendix A.

[2] An alternative approach is to apply the Lie derivative with respect to V to equation (3.1), making use of formula (3.2) of [22]: $L_V \mathrm{Ric}_h = -(n-2)\mathrm{Hess}_h \sigma - \Delta\sigma \cdot h$.

Proposition 5.2 *A vector field V satisfies*

$$L_V h = 2\sigma h,$$

with h given by (5.1) if and only if

(1) *V_t is a conformal field for g_N for each t with $L_{V_t} g_N = 2\omega_t g_N$,*
(2) $\frac{\partial}{\partial t}(\frac{v_0}{\sigma'}) = \frac{\omega_t}{\sigma'}$, *and*
(3) $\frac{\partial V_t}{\partial t} = -\frac{1}{(\sigma')^2} \nabla^N v_0$.

Moreover, $\sigma = \frac{v_0 \sigma''}{\sigma'} + \omega_t = \frac{\partial v_0}{\partial t}$.

We consider separately the cases in which $f = f(t)$ and $f = f(x)$, with the goal of classifying the structures of g and V, both locally and globally.

5.1 $f = f(t)$ Case

The first observation is that f is constant when $c = 0$.

Proposition 5.3 *If $f = f(t)$ and $D_V f = 0$, then f is constant.*

Proof Suppose I is an open interval on which $f'(t) \neq 0$. The condition $D_V f = 0$ is equivalent to $v_0(t, x) f'(t) = 0$, so $v_0 = 0$ on I. From Proposition 5.2, $\sigma = \frac{\partial v_0}{\partial t} = 0$ on I. This contradicts the fact that the zeros of σ are isolated. □

The following corollary is a special case.

Corollary 5.4 *Suppose σ has a critical point at $p \in M$. Then f is constant on any polar coordinate neighborhood of p.*

Proof If $d\sigma(p) = 0$, then h admits polar coordinates about p and $f = f(t)$. By smoothness, $df(p) = 0$. Since $D_V f$ is constant, it is identically zero. □

Now we may prove Theorem 1.7 from the Introduction, restated below for the reader's convenience.

Theorem 5.5 *Suppose $(M, g, f, \alpha, \lambda)$ is a complete GQE manifold, with $\alpha \neq \frac{1}{n-2}$, that admits a structure-preserving non-homothetic conformal field: $L_V g = 2\eta g$. If η has a critical point (e.g., if M is compact), then f is constant and (M, g) is isometric to a simply connected space form.*

Proof (M, h) admits a polar coordinate neighborhood U about a critical point p of σ with $f = f(t)$ on U, so that g is rotationally symmetric with pole p. By Corollary 5.4, f is constant on U. In the compact case, U covers M except for a point, so f is constant; in the non-compact case, $U = M$, and f is constant. Thus g is Einstein. Complete, rotationally symmetric Einstein manifolds are well known to be the simply connected space forms. □

Thus, we restrict to the case in which M is non-compact and σ has no critical points; from the previous results, we may also assume $c \neq 0$, so that f' never vanishes. We assume h is of the form (5.1) on $U = I \times N$, and where $\sigma' > 0$ on $I = (a, b)$. We have that $v_0 = \frac{c}{f'(t)}$ and in particular, v_0 is a function of only t and never vanishes. Corollaries A.2 and A.3 imply that V_t is independent of t, $\omega := \omega_t$ is constant, and v_0 and σ solve

$$\sigma = \sigma''\left(A + \omega \int \frac{dt}{\sigma'}\right) + \omega$$

$$v_0 = \sigma'\left(A + \omega \int \frac{dt}{\sigma'}\right)$$

for some constant A. Defining $r(t) = \int \frac{dt}{\sigma'}$, an increasing function of t, these equations become

$$\sigma = \sigma''(A + \omega r) + \omega \tag{5.2}$$

$$v_0 = \sigma'(A + \omega r). \tag{5.3}$$

Note that we have not yet used the GQE structure; doing so yields the following.

Lemma 5.6 *Suppose (U, h), V, and σ are as above. If $(U, h, f, \alpha, \lambda)$ is a GQE manifold, V preserves the GQE structure, and $f = f(t)$, then*

(1) *σ is a solution to (5.2) for some constants A and ω,*
(2) *$V = v_0(t)\frac{\partial}{\partial t} + V_0$ where v_0 is given in terms of σ by (5.3) and is non-zero on (a, b), and V_0 is a fixed homothetic field for g_N with expansion factor ω,*
(3) *$f(t) = \int \frac{c}{v_0(t)} dt$, and*
(4) *$\alpha = K_1 + K_2\mu(x)$ where K_i is are explicit constants determined by A, ω, σ, c, and n (see (5.6)), and $\mathrm{Ric}_{g_N} = \mu(x) g_N$.*

Conversely, if $A, \omega, \sigma, V, f, \alpha, g_N, K_1, K_2$, and c satisfy (1)–(4), then $(U, h, f, \alpha, \lambda)$ is a GQE manifold with structure-preserving conformal field V.

Remark 5.7 In particular, α is constant if $n > 3$. The proof will also show that α is constant if $n = 3$ and $\omega \neq 0$.

We have already established (1)–(3) above. Before proving (4), we note the following fundamental fact about solutions to (5.2).

Proposition 5.8 *A function σ solves (5.2) if and only if the quantity*

$$K = (A + \omega r)\left(\sigma'\right)^2 - \sigma(\sigma - \omega) \tag{5.4}$$

is constant.

Remark 5.9 When $\omega = 0$ this is the well-known fact that $A(\sigma')^2 - \sigma^2$ is constant for solutions to $\sigma'' = \frac{\sigma}{A}$.

Proof Differentiate with respect to t and use $\frac{dr}{dt} = \frac{1}{\sigma'}$:

$$\frac{d}{dt}\big((A+\omega r)(\sigma')^2 - \sigma(\sigma-\omega)\big) = \omega\frac{dr}{dt}(\sigma')^2 + 2(A+\omega r)\sigma'\sigma'' - 2\sigma\sigma' + \omega\sigma'$$
$$= 2\sigma'\big((A+\omega r)\sigma'' + \omega - \sigma\big). \qquad \square$$

Proof of Lemma 5.6 In order to have a GQE structure, the Ricci curvature of h must be given both from the warped product formula (3.4) and from (3.1), leading to

$$\left(\frac{1}{n-2} - \alpha\right) f'(t)^2 = -(n-2)\left(\frac{\sigma'''(t)}{\sigma'(t)} - \frac{\sigma''(t)^2}{\sigma'(t)^2}\right) - \frac{\mu(x)}{\sigma'(t)^2}. \tag{5.5}$$

We rewrite (5.2) as

$$\sigma'' = \frac{\sigma - \omega}{A + \omega r}.$$

Differentiating this equation with respect to t yields

$$\frac{\sigma'''}{\sigma'} = \frac{1}{A+\omega r} - \frac{\omega\sigma''}{(\sigma')^2(A+\omega r)}.$$

Substituting into formula (5.5) gives

$$\left(\frac{1}{n-2} - \alpha\right) f'(t)^2 = -(n-2)\left(\frac{\sigma'''}{\sigma'} - \frac{(\sigma'')^2}{(\sigma')^2}\right) - \frac{\mu(x)}{\sigma'(t)^2}$$
$$= -(n-2)\left(\frac{(\sigma')^2(A+\omega r) - \sigma(\sigma-\omega)}{(\sigma')^2(A+\omega r)^2}\right) - \frac{\mu(x)}{\sigma'(t)^2}.$$

On the other hand,

$$f'(t) = \frac{c}{v_0} = \frac{c}{\sigma'(A+\omega r)},$$

and $(A+\omega r)(\sigma')^2 - \sigma(\sigma-\omega) = K$ is constant, implying

$$\alpha = \frac{1}{n-2} + \frac{(n-2)K + \mu(x)(A+\omega r)^2}{c^2}.$$

However, if $\omega \neq 0$, in order for g_N to admit a non-Killing homothetic field, it must be flat (cf. p. 242 of [18]). Therefore, we have

$$\alpha = \begin{cases} \frac{1}{n-2} + \frac{(n-2)K + \mu(x)A^2}{c^2}, & \omega = 0 \\ \frac{1}{n-2} + \frac{(n-2)K}{c^2}, & \omega \neq 0. \end{cases} \tag{5.6}$$

$\square$

We separately analyze the cases in which $\omega = 0$ and $\omega \neq 0$. If $\omega = 0$, then $A \neq 0$, and the possible solutions to (5.2) are (up to shifting t and rescaling V and σ): $\sigma(t) = \cos(\kappa t)$, $\sigma(t) = e^{\kappa t}$, $\sigma(t) = \sinh(\kappa t)$, or $\sigma(t) = \cosh(\kappa t)$, where $\kappa = \sqrt{\frac{1}{|A|}}$. These all produce local examples.

We are interested in determining when it is possible to construct an example with g complete. To simplify notation, we assume $\kappa = 1$.

Example 5.10 Suppose $\sigma(t) = \sinh(t)$, and

$$h = dt^2 + \cosh(t)^2 g_N,$$

$$f = \int_0^t \frac{dr}{\cosh(r)},$$

$$V = \cosh(t)\frac{\partial}{\partial t} + X,$$

where N is any complete space with $\mathrm{Ric}_{g_N} = \mu g_N$, with Killing field X (possibly zero). One can readily check that V is a conformal field for $g = e^{\frac{2f}{n-2}} h$ (with $\omega = 0$) with expansion factor $\eta = \sinh(t) + \frac{1}{n-2}$, that $D_V f = 1$, and that g is complete (since f is bounded). Moreover, choosing α so that

$$\alpha = \frac{1}{n-2} + n - 2 + \mu$$

assures that $(M, g, f, \alpha, \lambda)$ is a GQE manifold for some λ.

Example 5.11 A similar example occurs with $\cosh(t)$ replaced with e^t and

$$\alpha = \frac{1}{n-2} + \mu.$$

However, in this case, f is given by $-e^{-t}$ (up to a constant), and the conformal metric $g = e^{\frac{2f}{n-2}} h$ is necessarily incomplete.

Example 5.12 Suppose $\sigma(t) = \cosh(t)$, so that σ has a critical point at $t = 0$. Then h is defined only on $(0, \infty)$ (or its negative), and the arc length with respect to g is given up to constants by

$$s(t) = \int_1^t \exp\left(\frac{c}{n-2}\int_1^z \frac{dy}{\sinh(y)}\right) dz.$$

However, $\lim_{t\to 0^-} s(t)$ is finite, so that g is incomplete. A similar argument applies if $\sigma(t) = \cos(t)$.

Next, we move on to the case in which $\omega \neq 0$. Perform the change of variables $r = \int_0^t \frac{dt}{\sigma'(t)}$. Since $\frac{d\sigma}{dr} = (\frac{d\sigma}{dt})^2$, (5.4) becomes

$$\frac{d\sigma}{dr} = \frac{K + \sigma(\sigma - \omega)}{A + \omega r}.$$

Separating variables and completing the square produces:

$$\int \frac{d\sigma}{K - \frac{\omega^2}{4} + (\sigma - \frac{\omega}{2})^2} = \int \frac{dr}{A + \omega r} = \frac{1}{\omega} \ln |C(A + \omega r)|,$$

for some constant $C > 0$. Let $B = K - \frac{\omega^2}{4}$. There are three cases depending on the sign of B.

- If $B = 0$ then $\sigma(r) - \frac{\omega}{2} = \frac{-\omega}{\ln|C(A+\omega r)|}$.
- If $B > 0$ then $\sigma(r) - \frac{\omega}{2} = \sqrt{B}\tan(\sqrt{B}\ln|C(A+\omega r)|)$.
- If $B < 0$ then

$$\begin{aligned}\sigma(r) - \frac{\omega}{2} &= \sqrt{-B}\tanh\bigl(\sqrt{-B}\ln|C(A+\omega r)|\bigr) \\ &= \sqrt{-B}\frac{|C(A+\omega r)|^{2\sqrt{-B}} - 1}{|C(A+\omega r)|^{2\sqrt{-B}} + 1}.\end{aligned}$$

Computing the derivative of f with respect to r using $f'(t) = \frac{c}{v_0(t)}$ and (5.3) gives

$$\frac{df}{dr} = \frac{c}{A + \omega r}.$$

So $f(r) = c\ln(D|A+\omega r|)$ for a constant $D > 0$.

Thus, we have completely determined the local structure of g, f, and V in the $\omega \neq 0$ case. Conversely, given constants $\omega \neq 0$, $c \neq 0$, $C > 0$, $D > 0$, A, B, we can use the above formulas for $f(r)$ and $\sigma(r)$ to construct local examples; the parameter t may be recovered by $t(r) = \int \sqrt{\sigma'(r)}dr$. Next, we are interested in analyzing which of these examples is complete.

We begin with a function $\sigma(r)$ of one of the three forms above, defined on a maximal interval I such that $\frac{d\sigma}{dr} > 0$. To simplify calculations we assume that $\omega = 1$ by rescaling V and σ; $A = 0$ by shifting s; $D = 1$ by shifting f; and $r > 0$ by symmetry. In each of the following cases, $f(r) = c\ln(r)$.

- If $B = 0$, then

$$\begin{aligned}\sigma(r) &= \frac{1}{2} - \frac{1}{\ln(Cr)}, \\ \frac{d\sigma}{dr} &= \frac{1}{r\ln(Cr)^2}.\end{aligned}$$

  σ is undefined at $r = 1/C$, so we consider $I = (0, 1/C)$ or $(1/C, \infty)$. The arc-length parameter for g is given by

$$\begin{aligned}s(r) = \int e^{\frac{f}{n-2}}dt &= \int r^{\frac{c}{n-2}}\left(\frac{1}{r^{1/2}\ln(Cr)}\right)dr \\ &= \int \frac{r^{\frac{2(c+1)-n}{2(n-2)}}}{\ln(Cr)}dr.\end{aligned}$$

  By analyzing the limiting behavior of $s(r)$ at $r = 0^+, 1/C^{\pm}$, and ∞, we find that g is complete with $I = (0, 1/C)$ if and only if $c \leq -\frac{n-2}{2}$ and with $I = (1/C, \infty)$ if and only if $c \geq -\frac{n-2}{2}$.

- If $B > 0$, then

$$\sigma(r) = \frac{1}{2} + \sqrt{B}\tan\big(\sqrt{B}\ln(Cr)\big),$$
$$\frac{d\sigma}{dr} = \frac{B\sec^2(\sqrt{B}\ln(Cr))}{r}.$$

We take the interval $I = (\frac{1}{C}e^{\frac{-\pi/2+k}{\sqrt{B}}}, \frac{1}{C}e^{\frac{\pi/2+k}{\sqrt{B}}})$ for any integer k. t is given by

$$t = \int \frac{\sqrt{B}\sec(\sqrt{B}\ln(Cr))}{\sqrt{r}}dr,$$

which implies that t is defined on $(-\infty, \infty)$. Since f is bounded in this case, g is complete.

- If $B < 0$, then

$$\sigma(r) = \frac{1}{2} + \sqrt{-B}\frac{|Cr|^{2\sqrt{-B}} - 1}{|Cr|^{2\sqrt{-B}} + 1},$$
$$\frac{d\sigma}{dr} = \frac{-4B|Cr|^{2\sqrt{-B}-1}}{(|Cr|^{2\sqrt{-B}} + 1)^2},$$

and we take $I = (0, \infty)$. The arc length with respect to g is:

$$s(r) = \int \frac{2\sqrt{-B}r^{\frac{c}{n-2}}(Cr)^{\sqrt{-B}-1/2}}{(Cr)^{2\sqrt{-B}} + 1}. \tag{5.7}$$

For no values of $B < 0, C > 0, c \neq 0$ does $|s(r)|$ limit to infinity at $r = 0$ and $r = \infty$; thus metrics of this form are incomplete.

At this point we have a full understanding of the $f = f(t)$ case, in both the $\omega = 0, \omega \neq 0$ subcases. For future reference, we analyze the completeness of the conformal field V.

Lemma 5.13 *If (M, g) is complete and non-compact in the $f = f(t)$ case, then the conformal field V is not complete.*

Proof Suppose V is complete. By Lemma 5.6, V is of the form $V = v_0(t)\frac{\partial}{\partial t} + V_0$, where V_0 is a fixed homothetic field for g_N. If g is complete, so is g_N. It follows that V_0 is a complete field (see p. 234 of [18]) on N and extends naturally to a complete vector field on M. Then $V - V_0 = v_0(t)\frac{\partial}{\partial t}$ is a complete field on M and therefore on $\mathbb{R}$. We analyze the two cases.

If $\omega = 0$, then $v_0(t)\frac{\partial}{\partial t} = \cosh(t)\frac{\partial}{\partial t}$, which is not a complete field on $\mathbb{R}$ by elementary considerations. This is a contradiction.

If $\omega \neq 0$, we can write $v_0\frac{\partial}{\partial t}$ as $(A + \omega r)\frac{\partial}{\partial r} = r\frac{\partial}{\partial r}$. The flow of this vector field at time ϵ is given by scaling r by e^{ϵ}. Such flows are globally well defined only on $\mathbb{R}$, $(-\infty, 0)$, and $(0, \infty)$. None of the complete examples we considered above were defined on such a subset, again leading to a contradiction. □

5.2 $f = f(x)$ Case

We also analyze the case of a conformal field in the $f = f(x)$ setting, so that $n \geq 4$ and Q is constant (by Observation 5.1 and Theorem 1.2). Without loss of generality, assume f is non-constant. We prove:

Proposition 5.14 *If the $f = f(x)$ case occurs, then σ solves*

$$\sigma''' = -\frac{Q}{n-1}\sigma'. \tag{5.8}$$

Moreover, $D_{V_t} f = c$ and $D_{V_t}\alpha = 0$, and either:

(1) *V_t is a non-homothetic conformal field on g_N for some t, or else*
(2) *V_t is a Killing field on g_N, independent of t, $v_0 = v_0(t)$ is a constant multiple of $\sigma'(t)$, and $\sigma'(t)$ is non-constant.*

Examples of case (2) are found using Sect. 4.2. After the proof, we demonstrate that case (1) may occur as well.

Proof First, by Lemma 3.8 and Observation 5.1, σ satisfies (5.8) and $\alpha = \alpha(x)$. Additionally, since $D_V f = c$ and $f = f(x)$, we have $D_{V_t} f = c$ for each V_t, and likewise $D_{V_t}\alpha = D_V\alpha = 0$.

By Proposition 5.2, for each t, V_t is a conformal field with expansion factor $\omega_t(x)$ on N. If any $\omega_t(\cdot)$ is non-constant on N, V_t is non-homothetic on N, and we are in case (1).

Otherwise, ω depends only on t. By (5.8), if σ' is constant, then $Q = 0$. If $\omega_t \neq 0$ for some t, then g_N admits a homothetic field that is non-isometric and so g_N is flat. Combining (3.1) and (3.4) shows that $(\frac{1}{n-2} - \alpha)df \otimes df$ is pointwise proportional to g_N. By comparing rank, it follows (since $\alpha \neq \frac{1}{n-2}$) that f is constant, a contradiction. We conclude that ω_t is identically zero. But Proposition 5.2 and the constancy of σ' imply $\sigma \equiv 0$, a contradiction.

Thus, we may assume σ' is non-constant, so that $\sigma''(t)$ vanishes only for isolated t by (5.8). From $\sigma = \frac{v_0}{\sigma'}\sigma'' + \omega_t$ of Proposition 5.2, we see that $v_0 = v_0(t)$. Then by Corollary A.2, V_t is independent of t and $\omega_t = \omega$ is constant in t and x. If $\omega \neq 0$, g_N admits a homothetic field that is not Killing, and the same argument as above leads to a contradiction. Thus $\omega = 0$, and Proposition 5.2 implies v_0 is a constant multiple of $\sigma'(t)$. $\square$

We demonstrate that the first case of Proposition 5.14 can occur, at least locally.

Example 5.15 Suppose that (K, h_K) is some n-manifold ($n \geq 3$) satisfying

$$\mathrm{Ric}_{h_K} = \left(\frac{1}{n-2} - \alpha_K\right)df \otimes df - (n-1)h_K$$

for a non-constant function $f : K \to \mathbb{R}$ and some constant α_K (cf. Example 4.8). Define $M = \mathbb{R}^2 \times K$ with coordinates (t, y) on $\mathbb{R}^2$ and metric

$$h_M = dt^2 + \cosh^2 t\left(dy^2 + \cosh^2 y\, h_K\right),$$

which satisfies

$$\mathrm{Ric}_{h_M} = \left(\frac{1}{n} - \alpha_M\right) df \otimes df - (n+1) h_M$$

for an appropriate constant α_M. Note the t-level sets each admit a conformal field $\cosh(y)\frac{\partial}{\partial y}$ with expansion factor $\sinh(y)$. We define a vector field V on M by

$$V = \left(\cosh(t)\sinh(y)\int_0^t \frac{\varphi(z)}{\cosh(z)} dz\right)\frac{\partial}{\partial t} + \varphi(t)\cosh(y)\frac{\partial}{\partial y},$$

where

$$\varphi(t) = \sin\big(2\arctan\big(\tanh(t/2)\big)\big).$$

Direct calculation (using Proposition A.1) shows that V is a conformal field of h with expansion factor $\sigma(t,y) = \sinh(y)(\sinh(t)\int_0^t \frac{\varphi(y)}{\cosh(y)} dr + \varphi(t))$; its restriction V_t to a level set of t is a conformal field with expansion factor $\omega_t(y) = \varphi(t)\sinh(y)$; in particular, V_t is non-homothetic for almost all $t \in \mathbb{R}$.

Moreover, V preserves the GQE structure of $(M, e^{\frac{2f}{n}} h_M)$: $D_V f = 0$, since f is a function on K, and $D_V \alpha_M = 0$ since α_M is constant.

Remark 5.16 In the above example, h_M admits a warped product structure with respect to the level sets of σ, by our classification theorem. However, we point out this structure is not apparent from the expression of h_M in coordinates t, y.

5.3 Complete Conformal Fields

Here we prove the generalization of the theorem of Yano and Nagano on complete conformal fields on Einstein spaces stated in the Introduction.

Proof of Theorem 1.8 Suppose $(M, g, f, \alpha, \lambda)$ is a complete GQE manifold equipped with a structure-preserving non-homothetic conformal vector field V: $L_V g = 2\eta g$. Assume V is complete. If η has a critical point, then by Theorem 5.5, (M, g) is a space form and f is constant. However, the round sphere is the only space form admitting a complete non-homothetic conformal field.

Otherwise, M is non-compact, $\sigma = \eta - \frac{c}{n-2}$ has no critical points, and the work of Sects. 5.1 and 5.2 applies. If $f = f(t)$, then Lemma 5.13 implies that V is incomplete, a contradiction. Thus $f = f(x)$ on M.

In this case, since g is complete, Theorem 1.2 and Observation 5.1 imply that h is a one-dimensional warped product defined for all $t \in \mathbb{R}$. Since σ' solves (5.8) and has no zeros we conclude that $\sigma'(t)$ is (up to an overall scaling of σ and of V, and a translation of t) equal to 1, $e^{\kappa t}$, or $\cosh(\kappa t)$, where $\kappa = \sqrt{\frac{-Q}{n-1}}$. If $\sigma' \equiv 1$, then by Proposition 5.2, $\sigma' = \omega_t$, which implies ω_t depends only on t. This contradicts part (2) of Proposition 5.14.

On the other hand, the following argument, which is an adaptation of Yano–Nagano's argument in the Einstein case, shows that $\nabla\sigma$ must be a complete field if

V is complete. This is a contradiction, since $e^{\kappa t}\frac{\partial}{\partial t}$ and $\cosh(\kappa t)\frac{\partial}{\partial t}$ are not complete fields on $\mathbb{R}$.

Computing the Laplacian on the warped product h gives

$$\begin{aligned} L_{\nabla\sigma}h &= 2\frac{\Delta\sigma}{n}h \\ &= 2\sigma'' h. \end{aligned}$$

However, by Proposition 5.14, $\sigma'' = -\frac{Q}{n-1}\sigma + c_0$, where Q and c_0 are constants. In particular, $W = \frac{Q}{n-1}V + \nabla\sigma$ satisfies $L_W h = 2c_0 h$. In the metric g,

$$L_W g = \left(\frac{2D_W f}{n-2} + c_0\right)g.$$

However, $D_W f$ is constant, since $D_V f$ is constant and $\nabla\sigma$ is orthogonal to ∇f. It follows that W is a homothetic field for the complete metric g, and so W is complete (see p. 234 of [18]). Since the set of complete conformal fields on a Riemannian manifold forms a Lie algebra, $\nabla\sigma$ is complete, a contradiction to the form of σ. $\square$

6 Gradient Ricci Solitons and *m*-Quasi-Einstein Metrics

In this section we specialize to the case where α and λ are constant, first obtaining some rigidity for the function Q of Proposition 3.1.

Proposition 6.1 *If (M, g, f) is a complete gradient Ricci soliton or a complete m-quasi-Einstein manifold, then Q is constant if and only if f is constant.*

Proof First note that if f is constant, the same is true for Q by definition. Now we prove the converse.

Gradient Ricci soliton case: If g is a gradient Ricci soliton ($\alpha = 0$ and λ constant) we have the following formula due to Hamilton (see Proposition 1.15 of [11] for a proof):

$$\Delta f - |\nabla f|^2 = -2\lambda f + c, \tag{6.1}$$

for some constant c. Plugging into the formula for Q (3.3), we obtain

$$Q = \frac{1}{n-2}\big(-2\lambda f + c + (n-2)\lambda\big)e^{\frac{2f}{n-2}}.$$

From this, one can see that if dQ vanishes identically and $df \neq 0$ at some point, then $\lambda = c = 0$. Then we have

$$\Delta f - |\nabla f|^2 = 0.$$

Moreover, $\Delta f = -R$ from the trace of the soliton equation, where R is the scalar curvature of g. In particular,

$$-R - |\nabla f|^2 = 0.$$

However, Chen has shown that if $\lambda = 0$ then $R \geq 0$ [10] (cf. [33, 34]), implying $R = |\nabla f| = 0$ when $c = 0$.

m-quasi-Einstein case: If g is m-quasi-Einstein ($m > 0$, $\alpha = \frac{-1}{m}$, and λ constant) we have the equation proven by Kim–Kim [17] that

$$\Delta f - |\nabla f|^2 = m\left(\lambda - \mu e^{\frac{2f}{m}}\right)$$

for some constant μ, which gives

$$Q = \frac{1}{n-2}\left((n+m-2)\lambda - \mu e^{\frac{2f}{m}}\right)e^{\frac{2f}{n-2}}.$$

If dQ vanishes identically and $df \neq 0$ at some point, then $\lambda = \mu = 0$. By a result of Case, f is constant [5]. $\square$

Corollary 6.2 *Suppose (M, g, f) is a complete gradient Ricci soliton or a complete m-quasi-Einstein manifold. If (M, g, f) admits a non-homothetic structure-preserving conformal diffeomorphism or conformal field, then only case* (1.2) *in Theorem* 1.2 *may occur.*

Proof In the $f = f(x)$ case, Q is constant by Lemma 3.8 (and Observation 5.1 in the case of a conformal field). Then f is constant, so we may say without loss of generality that $f = f(t)$. $\square$

When the constant λ is nonnegative, we also have the following.

Proposition 6.3 *If a complete gradient Ricci soliton or complete m-quasi-Einstein metric (M, g) of the form* (1.2),

$$g = ds^2 + v(s)^2 g_N,$$

with g_N Einstein has $\lambda \geq 0$, then either g is rotationally symmetric (on $\mathbb{R}^n$ or S^n), or v is constant and g is the product metric on $\mathbb{R} \times N$.

Proof The result follows from the work of various authors. The main observation is that a complete metric of type (I) (see Definition 2.1) of the form (1.2) contains a line in the s-direction: a geodesic defined on $(-\infty, \infty)$ that is minimizing on all its sub-segments.

In the m-quasi-Einstein case, a version of the Cheeger–Gromoll splitting theorem holds if $\lambda \geq 0$ [12]. Therefore, if g is not a product then it must be rotationally symmetric (i.e., type (II) or (III)). In fact, if $\lambda > 0$, M must be compact [30].

If (M, g) is a gradient Ricci soliton, we may, without loss of generality, replace g_N with a space form of the same dimension and with the same Einstein constant. In particular, (M, g) is now locally conformally flat. Locally conformally flat gradient Ricci solitons with $\lambda \geq 0$ are classified [4], however, we do not need the entire argument in this special case. Indeed, by the work of Chen [10] and Zhang [35] (cf. Proposition 2.4 of [4]) a locally conformally flat gradient Ricci soliton with $\lambda \geq 0$

either has positive curvature operator or is a product. However, by the classical splitting theorem of Toponogov, a space with positive curvature cannot contain a line, so a type (I) gradient Ricci soliton with $\lambda \geq 0$ must be a product. □

We now prove our main result on Ricci solitons stated in Sect. 1.

Proof of Theorem 1.9 The first claim that g_1 and g_2 are metrics of the form (1.2) follows from Theorem 1.2 and Corollary 6.2.

When g_1 is a complete shrinking or steady soliton, we also know from Proposition 6.3 that g_1 is either rotationally symmetric (on $\mathbb{R}^n$ or S^n) or a product. From the work of Kotschwar [19] and Bryant [3], the only complete rotationally symmetric gradient Ricci solitons with $\lambda_1 \geq 0$ are the round sphere, flat $\mathbb{R}^n$, and the Bryant soliton. In the flat case there are two rotationally symmetric gradient Ricci soliton structures with $f = f(s)$ on $g_1 = ds^2 + s^2 g_{S^{n-1}}$: one where f is constant and $\lambda_1 = 0$ and the other where f is the Gaussian density, $f = \frac{\lambda_1}{2}s^2 + b$. We will refer to the solitons in the first case as flat Euclidean solitons and to the second case as flat Gaussian solitons.

If g_1 is a product $\mathbb{R} \times N$ and $f_1 = f_1(t)$, we have that $\operatorname{Hess} f_1 = f_1'' dt^2$, so that g_N must be Einstein. By [15] any non-trivial compact gradient Ricci soliton is shrinking, so the compact result follows from the complete one. (In the trivial case in which f_1 is constant, we can appeal to Theorem 1.6.)

The next claim, that if g_2 is also a soliton, then both spaces are round spheres or ϕ is inverse stereographic projection, appears at the end of the section as Corollary 6.8.

Finally, we prove that a complete gradient Ricci soliton (M, g, f) admitting a non-homothetic, structure-preserving conformal field V is Einstein with f constant. If M is compact, then the first part of the proof, applied to the flow of V, implies that f is constant. Thus, we assume M is non-compact and f is non-constant and appeal to the classification derived in Sect. 5.1. Since we consider a complete Ricci soliton, only the $f = f(t)$ case occurs by Corollary 6.2. There are a couple cases to consider, in which f and σ are known explicitly.

Suppose $\omega = 0$. By translating t and rescaling σ, we have that $\sigma(t) = \frac{1}{\kappa}\sinh(\kappa t)$, $f' = \frac{c}{A\cosh(\kappa t)}$ for nonzero constants κ, c, and A, and

$$h = dt^2 + \cosh^2(\kappa t) g_N.$$

We compute $\operatorname{Ric}_h(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$ using both (3.2) and (3.4) to show that

$$\lambda = e^{-\frac{2f}{n-2}}\left(-\kappa^2(n-1) - \frac{1}{n-2}\big(f'\big)^2\right) - \frac{1}{n-2}\big(\Delta_g f - |\nabla f|_g^2\big),$$

where $g = e^{\frac{2f}{n-2}} h$ satisfies $\operatorname{Ric}_g + \operatorname{Hess}_g f = \lambda g$. Next, using the conformal relation between g and h, we find

$$\Delta_g f - |\nabla f|_g^2 = e^{-\frac{2f}{n-2}} \Delta_h f,$$

and, by computing the Laplacian on a warped product,

$$\Delta_h f = f'' + \frac{\kappa(n-1)f'\sinh(\kappa t)}{\cosh(\kappa t)}.$$

Thus,

$$\lambda = e^{-\frac{2f}{n-2}}\left(-\kappa^2(n-1) - \frac{1}{n-2}\left(f'' + \left(f'\right)^2 + \frac{\kappa(n-1)f'\sinh(\kappa t)}{\cosh(\kappa t)}\right)\right).$$

Elementary analysis shows that λ is non-constant.

Finally, suppose $\omega \neq 0$. In this case, g_N admits a homothetic field that is not Killing, so g_N is flat. Working in the variable $r = \int_0^t \frac{dt}{\sigma'(t)}$, we have

$$h = dt^2 + \sigma'(t)^2 g_N = \sigma'(t)^2\left(dr^2 + g_N\right).$$

The metric g is given by

$$g = e^{\frac{2f(r)}{n-2}}\sigma'(r)\left(dr^2 + g_N\right).$$

Let $\varphi = \frac{f}{n-2} + \frac{1}{2}\log\sigma'(r)$, so that $g = e^{2\varphi}(dr^2 + g_N)$. We use this conformal relation to find

$$\mathrm{Ric}_g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = -(n-1)\varphi''$$

$$\mathrm{Hess}_g f\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = f'' - \varphi' f',$$

where all derivatives are with respect to r. In particular, if $\mathrm{Ric}_g + \mathrm{Hess}_g f = \lambda g$, then

$$\lambda = -(n-1)\varphi'' + f'' - \varphi' f'.$$

Using $f'(r) = \frac{c}{r}$, straightforward computations show

$$\lambda = -\frac{n-1}{2}\left(\frac{\sigma'''}{\sigma'} - \frac{(\sigma'')^2}{(\sigma')^2}\right) - \frac{c\sigma''}{2r\sigma'} - \frac{c(c-1)}{(n-2)r^2}.$$

If $B = 0$, we have $\sigma(r) = \frac{1}{2} - \frac{1}{\ln(Cr)}$. If $B > 0$, then $\sigma(r) = \frac{1}{2} + \sqrt{B}\tan(\sqrt{B}\ln(Cr))$. Elementary analysis shows that λ is not constant in either case.

We conclude that f must in fact be constant, so that (M, g) is Einstein. $\square$

Remark 6.4 As an addendum to the proof of the last statement: Kanai showed a complete Einstein space admitting a non-homothetic conformal field belongs to the following list, up to rescaling [16] (cf. Theorem 2.7 of [22]): a round sphere, Euclidean space, hyperbolic space, a warped product $ds^2 + e^{2s}g_N$ (where N is Ricci-flat), or a warped product $ds^2 + \cosh^2(s)g_N$ (where N has Einstein constant $-(n-2)$).

We close with examples of Ricci solitons g_1 admitting structure-preserving conformal changes to GQE metrics g_2, making use of Corollary 4.10. These examples will be used in the proof of Corollary 6.8.

Example 6.5 Product soliton We consider the case in which $g_1 = ds^2 + g_N$, where g_N is Einstein with Einstein constant λ_1. $f_1 = f_1(s)$ is necessarily of the form

$$f_1(s) = \frac{\lambda_1}{2}s^2 + as + b,$$

for some constants a and b. Assume f_1 is not constant.

First we consider the case $\lambda_1 = 0$. By adding a constant to f_1, we may assume $f_1(s) = as$, $a \neq 0$. Then we have

$$h_1 = e^{\frac{-2as}{n-2}} g_1 = dt^2 + e^{\frac{-2as}{n-2}} g_N,$$

where $dt = e^{\frac{-as}{n-2}} ds$. u is a solution to $\frac{du}{dt} = e^{\frac{-as}{n-2}}$, which implies that $\frac{du}{ds} = e^{\frac{-2as}{n-2}}$. So $u(s) = -\frac{n-2}{2a} e^{\frac{-2as}{n-2}} + C$ and we have

$$g_2 = Ku^{-2} g_1 = K\left(d\tau^2 + u^{-2} g_N\right),$$

where K is a positive constant and $\tau(s) = \int u(s)^{-1} ds$.

If a and C have different signs, then $|u| > 0$ for all s, giving a global conformal change $g_2 = Ku^{-2} g_1$. Note that τ is always either bounded above or below (depending on the sign of a), so g_2 is not complete. If a and C have the same sign, then u has a zero, and the conformal change is not global.

In the case $\lambda_1 \neq 0$, by shifting s and adding a constant to f_1 we can assume that $f_1(s) = \frac{\lambda_1}{2} s^2$. Then we have

$$h_1 = e^{\frac{-\lambda_1 s^2}{n-2}} g_1 = dt^2 + e^{\frac{-\lambda_1 s^2}{n-2}} g_N,$$

where $dt = e^{\frac{-\lambda_1 s^2}{2(n-2)}} ds$. We have $\frac{du}{ds} = e^{\frac{-\lambda_1 s^2}{n-2}}$, so that $u(s) = C + \int_0^s e^{\frac{-\lambda_1 p^2}{n-2}} dp$ and $g_2 = Ku^{-2} g_1$. By Corollary 4.10, g_2 is a gradient Ricci almost soliton with potential $f = f_1$.

Note that when $\lambda_1 > 0$, $u(s)$ is bounded which implies that we can choose C large enough so that u does not vanish and that g_2 is complete if g_1 is. When $\lambda_1 < 0$, u will always have a zero, so there is no global conformal change.

In the case $\lambda_1 \geq 0$, we point out that (g_2, f) is not a gradient Ricci soliton. To see this, we note $\mathrm{Hess}_{g_1} u = u''(s) ds^2$ and $\Delta_{g_1} u = u''(s)$ and compute (where prime denotes a derivative with respect to s):

$$\mathrm{Ric}_{g_2} = (n-1)\left(\frac{u''}{u} - \frac{(u')^2}{u^2}\right) ds^2 + \left(\lambda_1 + \frac{u''}{u} - (n-1)\frac{(u')^2}{u^2}\right) g_N$$

and

$$\mathrm{Hess}_{g_2} f = \left(f'' + \frac{f'u'}{u}\right) ds^2 - \frac{f'u'}{u} g_N.$$

In order for $\mathrm{Ric}_{g_2} + \mathrm{Hess}_{g_2} f$ to equal $\lambda_2 g_2$, we must have

$$\lambda_2 = (n-1)\big(uu'' - (u')^2\big) + \big(\lambda_1 u^2 + \lambda_1 s u u'\big)$$

in the case $\lambda_1 > 0$, and

$$\lambda_2 = (n-1)\big(uu'' - (u')^2\big) + a u u'$$

in the case $\lambda_1 = 0$. However, in either case, one can explicitly show that λ_2 is not constant.

Example 6.6 (Bryant Soliton) The Bryant soliton is the unique (up to rescaling) complete, rotationally symmetric, steady, gradient Ricci soliton. We write this metric as

$$g_1 = ds^2 + w(s)^2 g_{S^{n-1}}$$

for $s \geq 0$, where $w(0) = 0$, $w'(0) = 1$, and $w(s) > 0$ for $s > 0$. We also have $w = O(s^{1/2})$, $w' = O(s^{-1/2})$, $w'' = O(s^{-3/2})$, the scalar curvature R is $O(s^{-1})$ for s large, and the sectional curvature is everywhere positive (see [3] or Chap. 1, Sect. 4 of [11]). From (6.1) we have

$$R + |\nabla f|^2 = c$$

for some positive constant c. Thus, $f' \to \pm\sqrt{c}$ at infinity, so $f = O(s)$. Since g_1 has positive curvature, $\mathrm{Hess} f$ is negative-definite and we conclude $f' \to -\sqrt{c}$ at infinity. Now,

$$h_1 = dt^2 + \big(e^{\frac{-f}{n-2}} w\big)^2 g_{S^{n-1}},$$

where $\frac{dt}{ds} = e^{-\frac{f}{n-2}}$, so that

$$u(s) = C + \int_0^s e^{\frac{-2f(p)}{n-2}} w(p) dp.$$

Since $s \geq 0$, from the asymptotics of f and w we see that u blows up exponentially in s. Thus we have a global conformal change to an incomplete metric g_2, provided $C > 0$.

Finally, we ask whether $g_2 = u^{-2} g_1$ is also a Ricci soliton with potential f. Assume $\mathrm{Ric}_{g_2} + \mathrm{Hess}_{g_2} f = \lambda_2 g_2$. Direct calculation of the ds^2 component of this equation leads to

$$(n-1)\left(\frac{u''}{u} - \frac{w''}{w} + \frac{u'w'}{uw} - \frac{(u')^2}{u^2}\right) + f'' + \frac{f'u'}{u} = \lambda_2 u^{-2}, \tag{6.2}$$

where all derivatives are with respect to s. Using $\mathrm{Ric}_{g_1} + \mathrm{Hess}_{g_1} f = 0$, we have $f'' = (n-1)\frac{w''}{w}$. This simplification leads to:

$$(n-1)\big(uu'' + uu'w'w^{-1} - (u')^2\big) + uu'f' = \lambda_2.$$

We show λ_2 is not constant by examining its asymptotics at $s=0$ and $s \to \infty$. As $s \to 0^+$: f limits to 0 (without loss of generality), f' limits to 0, u limits to C, u' limits to zero, and u'' limits to 1. This implies $\lim_{s\to 0^+} \lambda_2(s) = 2C(n-1)$. On the other hand, by the above asymptotics on w and f, one can show that $\frac{u''}{u}$ and $\frac{u'}{u}$ limit to $\frac{4c}{(n-2)^2}$ and $\frac{2\sqrt{c}}{n-2}$ at infinity, respectively. It follows that $\lim_{s\to\infty} \lambda_2(s)u(s)^{-2} = \frac{-2c}{n-2}$, so that $\lambda_2(s) \to -\infty$ as $s \to \infty$. In particular, λ_2 is not constant.

Example 6.7 (Flat Gaussian Soliton) There is one more example of a complete, rotationally symmetric, shrinking gradient Ricci soliton: the flat Gaussian. In this case the metric is flat $\mathbb{R}^n$ written in polar coordinates as

$$g_1 = ds^2 + s^2 g_{S^{n-1}}$$

with $f = \frac{\lambda_1}{2}s^2 + b$, $\lambda_1 \neq 0$. Without loss of generality we assume $b=0$. Then we have

$$h_1 = dt^2 + \left(e^{\frac{-\lambda_1 s^2}{2(n-2)}} s\right)^2 g_{S^{n-1}},$$

where $\frac{dt}{ds} = e^{\frac{-\lambda_1 s^2}{2(n-2)}}$, so $\frac{du}{ds} = se^{\frac{-\lambda_1 s^2}{n-2}}$ and thus

$$u(s) = C - \frac{(n-2)}{2\lambda_1} e^{\frac{-\lambda_1 s^2}{n-2}}.$$

Considering $g_2 = u^{-2} g_1$, note that when $\lambda_1 > 0$, $u(s)$ is bounded which implies that we can choose C large enough so that u does not vanish and that g_2 is complete. When $\lambda_1 < 0$ and $C > 0$ we also obtain a global conformal change; however, g_2 will be incomplete.

Finally, we determine whether g_2 is a Ricci soliton. We know that $\mathrm{Ric}_{g_2} + \mathrm{Hess}_{g_2} f = \lambda_2 g_2$ for a function λ_2. Arguing as in the Bryant soliton example, by equation (6.2) we obtain

$$(n-1)\left(uu'' + u'us^{-1} - \left(u'\right)^2\right) + \lambda_1 + \lambda_1 su'u = \lambda_2.$$

Then one can explicitly show that λ_2 is not constant in this case as well.

Finally, we prove the following corollary, which completes the proof of Theorem 1.9.

Corollary 6.8 *Let ϕ be a non-homothetic conformal diffeomorphism between Ricci solitons (M_1, g_1, f_1) and (M_2, g_2, f_2) such that $\phi^* df_2 = df_1$. If (M_1, g_1) is complete and either shrinking or steady, then f_1 and f_2 are constant, and either (M_1, g_1) and (M_2, g_2) are both isometric to round spheres, or ϕ is an inverse stereographic projection with (M_1, g_1) flat Euclidean space and (M_2, g_2) a round spherical metric with a point removed.*

Proof By Theorem 1.9, we have that (M_1, g_1, f_1) is a product of $\mathbb{R}$ with an Einstein manifold, the Bryant soliton, a flat Gaussian soliton, a flat Euclidean space, or a round sphere. However, the previous examples show the conformal transformations associated with the first three cases do not produce a soliton metric. In the last two cases, f is constant so we are in the Einstein case. From Example 4.11 we can see the only time we have a global non-homothetic conformal diffeomorphism from a round spherical metric g_1 to another Einstein metric g_2 is when g_2 is also a round spherical metric. A similar analysis shows that the only time we have a global non-homothetic conformal diffeomorphism from flat Euclidean space g_1 to another Einstein metric is the case of inverse stereographic projection where g_2 is a round spherical metric with a point removed. □

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Appendix A: Conformal Fields on Warped Products over a One-Dimensional Base

Here we collect some calculations for conformal fields of a Riemannian metric h of the form

$$h = dt^2 + u(t)^2 g_N$$

on $M = I \times N$, where I is an open interval. Let V be a vector field on M. We write

$$V = v_0(t,x)\frac{\partial}{\partial t} + V_t,$$

where v_0 is a function on M and V_t is the projection of V onto the factor $\{t\} \times N$. We have the following necessary and sufficient conditions for V to be a conformal field for h.

Proposition A.1 *V is a conformal field for h,*

$$L_V h = 2\sigma h,$$

if and only if

(1) *V_t is a conformal field for g_N for each t: $L_{V_t} g_N = 2\omega_t g_N$;*
(2) $\frac{\partial}{\partial t}(v_0 u^{-1}) = \omega_t u^{-1}$;
(3) $\frac{\partial V_t}{\partial t} = -u^{-2}\nabla^N v_0$,

where $\nabla^N v_0$ is the gradient of $v_0(t,\cdot)$ on $\{t\} \times N$. Moreover,

$$\sigma = v_0 u^{-1}\frac{\partial u}{\partial t} + \omega_t = \frac{\partial v_0}{\partial t}.$$

Proof We compute the Lie derivative of h. Let $(x^1, \ldots, x^{n-1})$ be normal coordinates at some $p \in N$, and let $V_t = v_i(t,x)\partial_i$, with the Einstein summation convention in effect for $i = 1$ to $n - 1$. Here, $\partial_i = \frac{\partial}{\partial x^i}$ and we let $\partial_t = \frac{\partial}{\partial t}$. To begin, we record the following Lie brackets.

$$[V, \partial_t] = -\frac{\partial v_0}{\partial t}\partial_t - \frac{\partial v_i}{\partial t}\partial_i$$
$$[V, \partial_j] = -\frac{\partial v_0}{\partial x^j}\partial_t - \frac{\partial v_i}{\partial x^j}\partial_i.$$

Now, at the point (t, p),

$$\begin{aligned}
(L_V h)(\partial_t, \partial_t) &= D_V h(\partial_t, \partial_t) - 2h\big([V, \partial_t], \partial_t\big) \\
&= 2\frac{\partial v_0}{\partial_t}, \\
(L_V h)(\partial_t, \partial_j) &= D_V h(\partial_t, \partial_j) - h\big([V, \partial_t], \partial_j\big) - h\big(\partial_t, [V, \partial_j]\big) \\
&= u^2\frac{\partial v_j}{\partial t} + \frac{\partial v_0}{\partial x^j}, \\
(L_V h)(\partial_j, \partial_k) &= D_V h(\partial_j, \partial_k) - h\big([V, \partial_j], \partial_k\big) - h\big(\partial_j, [V, \partial_k]\big) \\
&= 2v_0uu'\delta_{jk} + u^2\left(\frac{\partial v_k}{\partial x^j} + \frac{\partial v_j}{\partial x^k}\right) \\
&= 2v_0uu'g_N(\partial_j, \partial_k) + u^2(L_{V_t}g_N)(\partial_j, \partial_k).
\end{aligned}$$

In particular, for arbitrary vector fields X, Y tangent to $\{t\} \times N$,

$$(L_V h)(\partial_t, X) = u^2 g_N\left(X, \frac{\partial V_t}{\partial t}\right) + D_X v_0,$$
$$(L_V h)(X, Y) = 2v_0uu'g_N(X, Y) + u^2(L_{V_t}g_N)(X, Y).$$

Then $L_V h$ equals $2\sigma h$ if and only if

$$\sigma = \frac{\partial v_0}{\partial t},$$
$$u^2\frac{\partial V_t}{\partial t} = -\nabla^N v_0,$$
$$L_{V_t}g_N = 2\omega_t g_N,$$

where $\omega_t := \sigma - v_0u^{-1}u'$. The first equation is equivalent to

$$\frac{\partial}{\partial t}\big(v_0u^{-1}\big) = \omega u^{-1}. \qquad \square$$

Two consequences of this result are the following.

Corollary A.2 *If V is a conformal field for h as above, then $v_0 = v_0(t)$ if and only if V_t is a fixed homothetic vector field for g_N.*

Proof Equation (3) of the previous proposition shows that $v_0 = v_0(t)$ if and only if V_t is independent of t. In this case, (2) implies that ω is constant. □

In fact, we can solve for v_0 and σ explicitly.

Corollary A.3 *With notation as above*,

$$v_0 = u(t)\left(A(x) + \int \frac{\omega_t(x)}{u(t)}dt\right)$$

$$\sigma = u'(t)\left(A(x) + \int \frac{\omega_t(x)}{u(t)}dt\right) + \omega_t(x),$$

where $A(x)$ is a function on N.

Proof Integrating equation (2) of the proposition with respect to t gives the formula for v_0. The formula for σ follows from $\sigma = \frac{\partial v_0}{\partial t}$. □

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